

# Matching Connectivity: On the Structure of Graphs with Perfect Matchings\*

Archontia C. Giannopoulou, Stephan Kreutzer, Sebastian Wiederrecht<sup>†</sup>  
Technical University Berlin

## Abstract

We introduce the concept of *matching connectivity* as a notion of connectivity in graphs admitting perfect matchings which heavily relies on the structural properties of those matchings. We generalise a result of Robertson, Seymour and Thomas for bipartite graphs with perfect matchings (see [10]) in order to obtain a concept of *alternating paths* that turns out to be sufficient for the description of our connectivity parameter. We introduce some basic properties of matching connectivity and prove a Menger-type result for matching  $n$ -connected graphs.

Furthermore, we show that matching connectivity fills a gap in the investigation of *n-extendable graphs* and their connectivity properties. To be more precise we show that every  $n$ -extendable graph is matching  $n$ -connected and for the converse any matching  $(n + 1)$ -connected graph either is  $n$ -extendable, or belongs to a well described class of graphs: the *brace  $h$ -critical graphs*.

**Keywords.** Matchings, Extendability, Connectivity, Menger-type, Brace

## 1 Introduction

Let  $G$  denote a finite, simple, and undirected graph with vertex set  $V(G)$  and edge set  $E(G)$ . A *matching* of a graph  $G$  is a set  $M \subseteq E(G)$  such that no two edges in  $M$  share a common endpoint. If  $e = xy \in M$ ,  $e$  is said to *cover* the two vertices  $x$  and  $y$ , with  $\nu(G)$  we denote the size of a maximum matching in  $G$ . A matching  $M$  is called *perfect* if every vertex of  $G$  is covered by an edge of  $M$ . Hence, if a graph  $G$  has a perfect matching,  $\nu(G) = \frac{|V(G)|}{2}$ . We denote by  $\mathcal{M}(G)$  the set of all perfect matchings of a graph  $G$ .

A graph is called  *$k$ -factor critical*, if the deletion of any set of  $k$  vertices of the given graph results in a graph that has a perfect matching. For the values  $k = 1$  and  $k = 2$ , these properties are more commonly called *factor-critical* and *bicritical*, respectively. Factor-critical and bicritical graphs play a huge role in a canonical theory of graph decompositions in terms of their maximum matchings. For more on the decomposition theory, as well as matching theory in general, the reader is referred to [3].

Inspired by the importance of those properties, Plummer introduced the concept of *matching extensions* in 1980 (see [6]). A connected graph is called  *$n$ -extendable*, if  $|V(G)| \geq 2n + 2$  and every

---

\*The full version of this paper will be available on arxiv.org within the next days.

<sup>†</sup>Technische Universität Berlin, Berlin, Germany. email: firstname.lastname@tu-berlin.de. This work has been supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (ERC consolidator grant DISTRUCT, agreement No 648527).

matching of size  $n$  in  $G$  can be extended to, respectively is contained in, a perfect matching of  $G$ . Since its first appearance matching extendability has been the subject of many researchers' interest. For more information on  $n$ -extendable graphs see also the surveys [7] and [8].

Matching extendability of graphs, as well as the property of having a perfect matching in the first place, can be linked, in some sense, to certain connectivity properties. A first impression on the nature of those properties is given by the classical characterisation of graphs with perfect matchings by Tutte in 1947 [11]. Let  $c_0(G)$  denote the number of components of  $G$  that have an odd number of vertices.

**Theorem 1.1 (Tutte [11])** *A graph  $G$  has a perfect matching if and only if for every set  $S \subseteq V(G)$ ,  $c_0(G - S) \leq |S|$ .*

This result was later generalised to a description of  $n$ -extendable graphs by Qinglin, a result on which we will heavily rely in the last section of this work.

**Theorem 1.2 (Qinglin [9])** *Let  $n \geq 1$ . A graph  $G$  is  $n$ -extendable if and only if for all  $S \subseteq V(G)$ ,*

1.  $c_0(G - S) \leq |S|$  and
2.  $c_0(G - S) = |S| - 2h$ , ( $0 \leq h \leq n - 1$ ) implies  $\nu(G[S]) \leq h$ .

In terms of vertices,  $n$ -extendability implies high vertex connectivity (from now on called connectivity). Plummer showed in [6] that any  $n$ -extendable graph is  $(n + 1)$ -connected. Unfortunately, though, connectivity alone does not seem to be enough, since the converse does not hold in general. In terms of edges, a certain concept of connectivity, namely, the *cyclic edge-connectivity*, has proven to be somewhat useful. In particular, a set of edges  $C$  in a graph  $G$  is called a *cyclic edge-cut* if  $G - C$  contains at least two components each of which contains a cycle. The size of any smallest cyclic edge-cut is called the *cyclic edge-connectivity* of  $G$ . In [4] (see also [5]) it was shown that  $r$ -regular non-bipartite graphs with an even number of vertices and cyclic edge-connectivity at least  $r + 1$  must be bicritical, while  $r$ -regular bipartite graphs with cyclic edge-connectivity at least  $(n - 1)r + 1$  are  $n$ -extendable.

There seems to be a difference between bipartite and non-bipartite graphs in terms of perfect matchings and their structural properties. In fact the building blocks in which every graph with a perfect matching can be decomposed, as shown by Lovász et al. in [1] and [2], are the so called *bricks*, which are the 3-connected bicritical graphs and thus not bipartite, and the *braces*, the bipartite and 2-extendable graphs.

In the case of bipartite graphs, there is also a very strong relation between matching extendability and the strong connectivity of digraphs. Given any bipartite graph  $G$  and any perfect matching  $M \in \mathcal{M}(G)$ , there is a directed graph corresponding to the pair  $(G, M)$ .

**Definition 1.3** Let  $G = (U \cup W, E)$  be a bipartite graph and  $M \in \mathcal{M}(G)$  be a perfect matching of  $G$ . The  $M$ -direction,  $\mathcal{D}(G, M)$  of  $G$  is defined as follows (see also Figure 1). Let  $e_1, \dots, e_{|M|}$  be an arbitrary ordering of the edges of  $M$  with  $e_i = u_i w_i$ ,  $1 \leq i \leq |M|$  and  $u_i \in U, w_i \in W$ . Then

1.  $V(\mathcal{D}(G, M)) := \{v_1, \dots, v_{|M|}\}$  and
2.  $E(\mathcal{D}(G, M)) := \{(v_i, v_j) \mid u_i w_j \in E(G)\}$ .

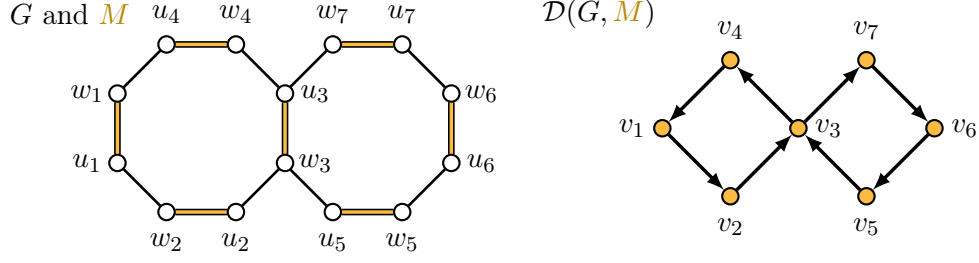


Figure 1: Graph bipartite  $G = (U \cup W, E)$  with perfect matching  $M$  and its  $M$ -direction.

Moreover, we will say that the edge  $v_i v_j \in E(\mathcal{D}(G, M))$  *mirrors* the edge  $u_i w_j \in E(G)$  and vice versa. In other words, the  $M$ -direction  $\mathcal{D}(G, M)$  of  $G$  is defined by orienting the edges of  $G$  that do not belong to  $M$  in such a way that the vertex in  $W$  is the head of the edge, and contracting the edges of  $M$ .

The following theorem, linking the strong connectivity of  $M$ -directions of bipartite graphs to matching extendability, is one of many results that have risen from the research of *Pfaffian orientations* and the problem of computing  $|\mathcal{M}(G)|$ .

**Theorem 1.4 (Robertson, Seymour and Thomas [10])** *Let  $G$  be a connected bipartite graph,  $n \geq 1$  and  $M \in \mathcal{M}(G)$ , then  $\mathcal{D}(M, G)$  is strongly  $n$ -connected if and only if  $\mathbf{cov}(G)$  is  $n$ -extendable.*

Extendability has a strong relation with multiple connectivity parameters, especially on bipartite graphs. Zhang and Zhang (see [12]) used another method to construct directed graphs out of a bipartite graph together with a perfect matching. Given a bipartite graph  $G = (U \cup W, E)$  and  $M \in \mathcal{M}(G)$ , we orient the edges in  $E$  as follows. Every edge in  $M$  is to be oriented towards its endpoint in  $W$ , while every edge in  $E \setminus M$  will be oriented towards its endpoint in  $U$ . The resulting directed graph, denoted by  $\vec{G}(M)$  is called the *residual graph* of  $G$  with respect to  $M$ .

Similar to Theorem 1.4, they linked the extendability of  $\mathbf{cov}(G)$  to some connectivity parameter of  $\vec{G}(M)$ . Let  $D$  be a directed graph with a pair of distinct vertices  $u$  and  $v$ . We call  $u$   *$n$ -arc connected* to  $v$  if there is a directed path from  $u$  to  $v$  in  $D$  after the removal of any set  $C \subseteq E$  with  $|C| \leq n - 1$ . The *arc connectivity* of  $D$  from  $u$  to  $v$ , denoted as  $\lambda(u, v)$ , is defined as the maximum integer  $n$ , such that  $u$  is  $n$ -arc connected to  $v$ .

Based on the 2-colouring of  $G$  we can now define a more local form of arc connectivity. Consider the bipartite graph  $G = (U \cup W, E)$  together with  $M \in \mathcal{M}(G)$  again. We define

$$\lambda^{WU}(\vec{G}(M)) := \min \{ \lambda(w, u) \mid u \in U \text{ and } w \in W \}.$$

**Theorem 1.5 (Zhang and Zhang [12])** *Let  $G = (U \cup W, E)$  be a bipartite graph and  $M \in \mathcal{M}(G)$ . Then  $\mathbf{cov}(G)$  is  $n$ -extendable if and only if  $\lambda^{WU}(\vec{G}(M)) \geq n$ .*

In this paper we propose a simple definition of what we call *matching connectivity* and present some basic results concerning this concept. While at first not concerning ourselves with paths for the definition itself, we will introduce a concept of alternating paths for a fixed perfect matching  $M$  that generalise the directed paths of the  $M$ -direction of bipartite graphs. In Section 2 we introduce

said parameter and present some structural results including a collection of characterisations of matching connectivity.

In Section 3 we generalize matching connectivity by introducing matching separators and characterising graphs with large matching separators by the number of their disjoint alternating paths. Finally, in Section 4, we apply the newly obtained concept of matching connectivity to describe  $n$ -extendable graphs and their properties.

## 2 Matching Connectivity

The goal of this section is to appropriately define connectivity in the context of matchings for the original graph  $G$  in such a way that it can be translated in a straight forward manner into strong connectivity of  $M$ -directions of bipartite graphs. First we need a few preliminary definitions that will be needed throughout the paper.

A graph  $G$  is called *matching covered*, if for every edge  $e \in E(G)$ , there is some  $M \in \mathcal{M}(G)$  with  $e \in M$ . The *cover graph*,  $\mathbf{cov}(G)$ , of  $G$  is the graph with vertex set  $V(G)$  that contains all edges  $e \in E(G)$  for which there is some  $M \in \mathcal{M}(G)$  with  $e \in M$ . Clearly,  $\mathbf{cov}(G)$  is matching covered and, in general, a graph  $G$  is matching covered if and only if  $G = \mathbf{cov}(G)$ .

A set  $S \subseteq V(G)$  of vertices is called *central* if  $G - S$  has a perfect matching. Given a matching  $M \in \mathcal{M}(G)$ , a set  $S \subseteq V(G)$  is called  *$M$ -central* if  $M$  is a perfect matching of both  $G - S$  and  $G[S]$ . Furthermore,  $S$  is called *strongly central* if both  $G - S$  and  $G[S]$  have a perfect matching.

An induced subgraph  $H \subseteq G$  is *central*, respectively  *$M$ -central* or *strongly central*, if  $V(H)$  is a central, respectively,  $M$ -central or strongly central, set, for some  $M \in \mathcal{M}(G)$ .

We call a cycle  $C \subseteq G$  an *alternating cycle* of  $G$  if there is a perfect matching  $M$  of  $G$  which also is a perfect matching of  $C$ . This cycle is also called  *$M$ -alternating*, for the aforementioned matching  $H$ . If  $C$  is  $M$ -alternating, clearly there is another perfect matching  $M' \neq M$  with  $E(C) \setminus M \subseteq M'$ . Hence, if needed,  $C$  will be called  *$M$ - $M'$ -alternating* to indicate that  $M$  and  $M'$  form a partition of the edges of  $C$ . A path  $P \subseteq G$  is called *alternating*, or  *$M$ -alternating*, if either  $M$  is a perfect matching of  $P$ , or there is an endpoint  $x \in V(P)$  such that  $M$  is a perfect matching of the subpath  $P - x$ .

**Lemma 2.1** *If  $G$  is a graph and  $H \subseteq G$  is a strongly central subgraph of  $G$  then  $\mathbf{cov}(H) \subseteq \mathbf{cov}(G)$ .*

PROOF. Suppose  $\mathbf{cov}(H)$  is not a subgraph of  $\mathbf{cov}(G)$ . Then there must be some edge  $e \in E(\mathbf{cov}(H))$  with  $e \in E(G) \setminus E(\mathbf{cov}(G))$ . So  $e$  is not contained in any perfect matching of  $G$ , but part of a perfect matching of  $H$ . Now, since  $H$  is strongly central, we know that both  $H$  and  $G - H$  have perfect matchings. Hence we can combine any perfect matching of  $H$  with any perfect matching of  $G - H$  and thereby obtain a perfect matching of the whole graph  $G$ . In particular by this method we can find a perfect matching of  $G$  containing  $e$  and thus contradicting its existence.  $\square$

**Definition 2.2** Let  $G$  be a graph. We call  $G$  *matching connected*, if  $\mathbf{cov}(G)$  is connected. If  $H \subseteq G$  is a maximal matching connected subgraph of  $G$ , we call  $H$  a *matching component* of  $G$ . If  $G$  consists of exactly two vertices joined together by an edge, we call it *trivial*.

**Observation 2.3** *A graph  $G$  is matching connected if and only if  $\mathbf{cov}(G)$  is matching connected.*

Notice that there exist connected graphs, in the usual graph-theoretic connectivity notion, which are not matching connected. As an example consider a path  $P$  of length  $2k + 1$  for some  $k \geq 1$ .

Such a path has exactly one perfect matching, say  $M$ . This matching contains both, the first and the last edge, as well as every second edge in between. The remaining edges can never be part of a perfect matching of  $P$  and thus  $\mathbf{cov}(P)$  contains exactly  $k + 1$  components, each of them consisting of exactly two vertices and an edge of  $M$  connecting them. For another example, see also Figure 2.

**Lemma 2.4** *If  $G$  is a non trivial matching connected graph, it is 2-connected.*

PROOF. Suppose that  $G$  contains more than two vertices and a cut vertex  $x \in V(G)$  such that  $G - x$  is not connected. Let  $C_1, \dots, C_\ell$ ,  $\ell \geq 2$ , be the components of  $G - x$ . Then, since  $G$  has a perfect matching,  $|V(G)|$  is even and thus  $|V(G - x)|$  is odd. So at least one of the components of  $G - x$ , say  $C_i$ , is odd. Moreover, there is no edge joining a vertex of  $C_i$  to any vertex of  $G - C_i - x$ .

Suppose there is some vertex  $y \in V(G) \setminus V(C_i)$  such that  $xy \in M$  for some perfect matching  $M \in \mathcal{M}(G)$ . By our assumption,  $C_i$  is an odd component of  $G - x - y$  and furthermore,  $M \setminus \{xy\}$  is a perfect matching of  $G - x - y$ . So in particular,  $M$  contains a perfect matching of  $C_i$  contradicting  $C_i$  to have an odd number of vertices.

Therefore, no edge between  $x$  and the components  $C_j \neq C_i$  can be contained in a perfect matching of  $G$ . Since  $G - x$  has at least two components, the graph  $\mathbf{cov}(G)$  is not connected, contradicting the matching connectivity of  $G$ . Thus the graph  $G$  is 2-connected.  $\square$

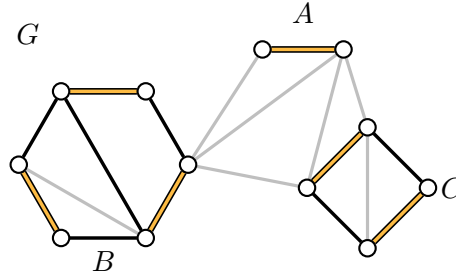


Figure 2: Graph  $G$  with perfect matching  $M$ , three different matching components (black and framed edges),  $A$ ,  $B$  and  $C$ , and edges, that do not belong to the cover graph. The matching components  $B$  and  $C$  are 2-connected. On the other hand the edge in  $A$  is not contained in an alternating cycle with any of the other edges of  $M$ . Within the non-trivial matching components  $B$  and  $C$ , any framed edge  $e$ , which thereby belongs to  $M$ , can be replaced. Replaced means that there is another perfect matching of the graph not containing  $e$ , but not touching the matchings within the other matching components.

**Lemma 2.5** *If  $C = (v_1, \dots, v_\ell)$  is an  $M$ -alternating cycle of  $G$  then there exists a perfect matching  $M' \in \mathcal{M}(G)$  such that  $M' \setminus E(C) = M \setminus E(C)$  and  $M' \cap E(C) = E(C) \setminus M$ .*

PROOF. The assertion follows from  $C$  being an  $M$ -alternating cycle, implying  $C$  to be an  $M$ -central subgraph of  $G$ . Hence we can construct another perfect matching  $M'$  such that  $M' \setminus E(C) = M \setminus E(C)$  and  $M' \cap E(C) = E(C) \setminus M$ .  $\square$

**Lemma 2.6** *Let  $G$  be a matching connected graph. Let also  $M \in \mathcal{M}(G)$  and  $xy \in M$ . For every perfect matching  $M' \in \mathcal{M}(G)$  with  $xy \notin M'$  there is an  $M$ - $M'$ -alternating cycle in  $G$  containing  $xy$ .*

PROOF. We may assume  $|V(G)| \geq 4$  otherwise  $G$  consists only of the edge  $xy$  and there is no perfect matching of  $G$  that does not contain  $xy$ . Let also  $M' \in \mathcal{M}(G)$  be any perfect matching not containing  $xy$ . Then, there exists  $e_x \in M'$  covering  $x$  and another vertex  $u \neq y$ . Let  $P$  be an  $M$ - $M'$ -alternating path of maximal length in  $\mathbf{cov}(G) - xy$  that contains  $u$  and has  $x$  as an endpoint. Notice that such a path exists since the edge  $xu$  is trivially such a path. Observe, if  $y$  is an internal vertex of  $P$  then, since  $xy \in M$ ,  $xy \in P$  follows. This is a contradiction to the choice of  $P \subseteq \mathbf{cov}(G) - xy$ . Thus, if  $y \in V(P)$ , then  $y$  is the other endpoint of  $P$ . Now let us assume  $y \notin V(P)$ . Let  $w$  denote the endpoint of  $P$  which is not  $x$  and  $e_w$  denote the edge of  $P$  containing  $w$ . Either  $e_w \in M$  or  $e_w \in M'$  holds, since  $P$  alternates between these two matchings.

We first exclude the case where  $e_w \in M$ . Suppose, towards a contradiction,  $e_w \in M$ . Since  $M'$  is a perfect matching of  $G$ , there exists an edge  $e'_w \in M'$  that covers  $w$ . Notice that all vertices of  $P - w$  are covered by the matching  $M'$ . In particular, since  $xy \in M$  and  $y \notin V(P)$ , the edge of  $P$  that contains  $x$  belongs to  $M'$ . This implies that the other endpoint of  $e'_w$  does not belong to  $P$ . Therefore, the path  $P \cup \{e'_w\}$  is an  $M$ - $M'$ -alternating path in  $\mathbf{cov}(G) - xy$  which contains  $u$ , has  $x$  as an endpoint and has one edge more than  $P$ , a contradiction to the maximal choice of  $P$ .

So now for  $e_w \in M'$ . Again aiming for a contradiction suppose  $e_w \in M'$ . As above, since  $M$  is a perfect matching of  $G$ , there exists an edge  $e'_w \in M$  which covers  $w$ . Notice that all internal vertices of  $P$  are covered by the matching  $M$ . Moreover  $xy \in M$  and  $xy \notin \mathbf{cov}(P) - xy$ . This implies that the other endpoint of  $e'_w$  does not belong to  $P$ . Therefore, as above, the path  $P \cup \{e'_w\}$  is an  $M$ - $M'$ -alternating path in  $\mathbf{cov}(G) - xy$  which contains  $u$  and has  $x$  as an endpoint and has one more edge than  $P$ , a contradiction to the maximality of the length of  $P$ .

So  $y \in V(P)$ . Hence  $y$  is the other endpoint of  $P$  and  $P \cup \{xy\}$  is the desired  $M$ - $M'$ -alternating-cycle, concluding the proof.  $\square$

## 2.1 Extendable Paths

As seen in the previous section, the notion of matching connectivity of a graph  $G$  is equivalent to the usual graph-theoretic notion of connectivity in the cover graph of  $G$ . But there seems to be more to it. The cover graph of a non-trivial matching connected graph  $G$  is always 2-connected. Our main goal is to show, if  $G$  is bipartite then the  $M$ -direction  $\mathcal{D}(G, M)$  is strongly connected, where  $M \in \mathcal{M}(G)$ . Working towards this next goal, we will first show, if a graph  $G$  is matching connected then, for any given perfect matching  $M \in \mathcal{M}(G)$ , there are paths that alternate between edges of  $M$  and edges not in  $M$ .

**Definition 2.7** Let  $G$  be a graph,  $x \in V(G)$ ,  $M \in \mathcal{M}(G)$ , and  $e = yy' \in M$ . Furthermore, let  $e' = xx' \in M$  be the edge of  $M$  covering  $x$ .

- i) An  $M$ -alternating path  $P$  with endpoints  $x$  and  $y$  or  $y'$  is called a *weakly extendable path respecting  $x$ ,  $e$  and  $M$*  or *weak  $x$ - $e$ - $M$ -extendable path* if it either consists exactly of the edge  $e$ , if  $e = e'$ , or does not contain  $e$  and  $e'$ , otherwise.
- ii) An  $M$ -alternating path  $P$  with endpoints  $x$  and  $y$  or  $y'$  is called a *strongly extendable path respecting  $x$ ,  $e$  and  $M$*  or *strong  $x$ - $e$ - $M$ -extendable path* if  $P$  is a *weak  $x$ - $e$ - $M$ -extendable path* and for every edge of  $f \in E(P) \setminus M$  there is some perfect matching  $M_f \in \mathcal{M}(G)$  with  $f \in M_f$ . Most of the time we will omit the word *strong* when we are talking about strongly extendable paths.

In order to get some intuition about the concept of extendable paths, let us revisit the graph of Figure 1 and its  $M$ -direction.

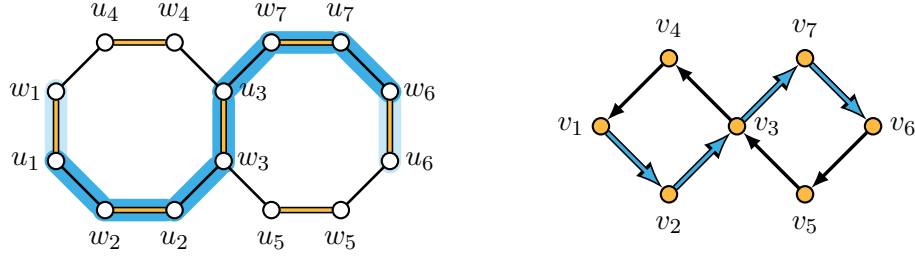


Figure 3: A directed path in the  $M$ -direction of  $G$  and its corresponding  $M$ -alternating paths in  $G$ . There are several possible such  $M$ -alternating paths that represent the **directed path**  $(v_1, v_2, v_3, v_7, v_8)$ . Such a path might start and end with an edge that does not belong to  $M$ , or it might include one or both such edges as the **first** or **last** one.

Consider the directed path  $P = (v_1, v_2, v_3, v_7, v_8)$  in the  $M$ -direction  $\mathcal{D}(G, M)$  of Figure 3. In the next lemma we will utilise the alternating paths in the uncontracted graph  $G$  to represent  $P$ . For this reason we want these paths to somehow preserve the orientation of  $P$  with respect to the colour classes  $U$  and  $W$ . For this reason we exclude the first and the last edge of  $M$ ,  $u_1w_1$  and  $u_6w_6$ , from our definition of extendable paths.

**Lemma 2.8** *Let  $G = (U \cup W, E)$  be a bipartite graph with bipartition  $(U, W)$ ,  $M \in \mathcal{M}(G)$  be a perfect matching,  $x \in U$ , and  $e \in M$ , with  $x \notin e$ . Furthermore, let  $e_x \in M$  be the edge covering  $x$ ,  $w \in e \cap W$ , and let  $v_{e_x}$  and  $v_e$  be the vertices of  $\mathcal{D}(G, M)$  corresponding to  $e_x$  and  $e$ . There is a weakly  $x$ - $e$ - $M$  extendable path  $P$  in  $G$  if and only if there is a directed path  $Q$  from  $v_{e_x}$  to  $v_e$  in  $\mathcal{D}(G, M)$ . The endpoints of this weakly extendable path are  $x$  and  $w$ . Moreover, the edges of  $P \setminus M$  in  $G$  mirror the edges of  $Q$  and vice versa.*

**PROOF.** We start with the reverse direction. So let  $P$  be a directed path in  $\mathcal{D}(G, M)$  starting with  $v_{e_x}$  and ending in  $v_e$ . Observe that in  $G$  every directed edge of  $P$  mirrors an undirected edge of  $G$  and every vertex of  $P$  corresponds to an edge of  $M$ . Let then  $v_s$  be a vertex of  $P$  and let  $e, e'$  be the edges of  $P$  containing  $v_s$  as head and tail respectively. Let  $e_s \in M$  be the matching edge of  $G$  corresponding to  $v_s$ . Notice then that, by definition, the endpoint corresponding to the head of  $e$  in  $G$  belongs to  $U$  and  $e_s$  and the endpoint corresponding to the tail of  $e'$  in  $G$  belongs to  $W$  and  $e_s$ . Note that in this way a directed path from  $v_{e_x}$  to  $v_e$  in  $\mathcal{D}(G, M)$  uniquely defines a weakly  $x$ - $e$ - $M$  extendable path in  $G$  with endpoints  $x$  and  $w$ .

For the straightforward direction consider any weak  $x$ - $e$ - $M$ -extendable path  $P'$ . First note that any weakly extendable path is of odd length, starting and ending with an edge not belonging to  $M$ . Since  $P'$  starts with a vertex of  $U$ , every second vertex of the path has to be in  $U$  as well and thus the other endpoint of every even length subpath of  $P'$  starting with  $x$  has to be in  $U$ . Hence the other endpoint of  $P'$  is exactly  $w \in e \cap W$ . Knowing this, it is easy to see that  $P'$  corresponds to a directed path starting in  $v_{e_x}$  and ending in  $v_e$  in  $\mathcal{D}(G, M)$ .  $\square$

From Lemma 2.8 we obtain that a bipartite graph  $G$  with a perfect matching is connected if and only if  $\mathcal{D}(G, M)$  is weakly connected for all  $M \in \mathcal{M}(G)$ . The following is an observation on the



edges of  $\mathcal{D}(G, M)$  mirroring edges of  $\mathbf{cov}(G) \setminus M$ . This result sheds light on the possible transitions from one  $M$ -direction to another  $M'$ -direction of  $G$ .

**Lemma 2.9** *Let  $G = (U \cup W, E)$  be a bipartite graph,  $M \in \mathcal{M}(G)$ , and  $e' \in E(\mathcal{D}(G, M))$ . If  $e \in E(G)$  is the edge of  $G$  that mirrors  $e'$ ,  $e \in E(\mathbf{cov}(G)) \setminus M$  if and only if  $e'$  is contained in a directed cycle in  $\mathcal{D}(G, M)$ .*

PROOF. Let  $e' = (x, y) \in E(\mathcal{D}(G, M))$  such that there is some directed cycle  $C'$  in the  $M$ -direction  $\mathcal{D}(G, M)$  containing  $e'$  and let  $v \neq y$  with  $v \in C'$ . Then consider the directed path  $P$  from  $y$  to  $v$  and the directed path  $Q$  from  $v$  to  $y$  in  $\mathcal{D}(G, M)$  induced by  $C'$ . Moreover, the paths  $P$  and  $Q$  are internally disjoint. Let  $e_y$  and  $e_v$  be the edges of  $M$  in  $G$  that correspond to the vertices  $y$  and  $v$ . From Lemma 2.8, there exist a weakly extendable path  $P_G$  from the endpoint of  $e_y$  in  $U$  to the endpoint of  $e_v$  in  $W$  (not containing the edges  $e_y$  and  $e_v$ ) and a weakly extendable path  $Q_G$  from the endpoint of  $e_v$  in  $U$  to the endpoint of  $e_y$  in  $W$  (not containing the edges  $e_y$  and  $e_v$ ). Notice then that  $P_G \cup Q_G \cup \{e_y, e_v\}$  form an  $M$ -alternating cycle  $C$  of  $G$  and  $e \notin M$  belongs to this cycle. Then from Lemma 2.5 there exists an  $M' \in \mathcal{M}(G)$  that contains  $E(C) \setminus M$ . Thus, since  $e \notin M$ ,  $e \in E(\mathbf{cov}(G))$  and, in particular,  $e \in E(\mathbf{cov}(G)) \setminus M$ .

For the reverse direction take any edge  $e \in E(\mathbf{cov}(G)) \setminus M$ , then there exists a matching  $M'$  with  $e \in M'$ . By Lemma 2.6 this edge is contained in an  $M$ - $M'$ -alternating cycle  $C$ . Then, contracting the edges of  $M$  for the construction of  $\mathcal{D}(G, M)$  halves the length of  $C$  and leaves a directed cycle  $C'$  in the  $M$ -direction  $\mathcal{D}(G, M)$  that contains  $e'$ .  $\square$

Next we will discuss some basic properties of (strongly) extendable paths, starting with their (major) role in matching connectivity.

**Theorem 2.10** *A graph  $G$  is matching connected if and only if for every  $x \in V(G)$ , every  $M \in \mathcal{M}(G)$ , and every  $e \in M$  there is a strong  $x$ - $e$ - $M$ -extendable path.*

PROOF. We begin with the assumption that there is a strong  $x$ - $e$ - $M$ -extendable path for every  $x \in V(G)$ ,  $M \in \mathcal{M}(G)$ , and  $e \in M$ . Since these paths are strong, every edge of the path belongs to some perfect matching of  $G$ . Therefore, each such path is also a path in the cover graph of  $G$ . Hence  $\mathbf{cov}(G)$  is connected and thus  $G$  is matching connected.

So now let us assume that  $G$  is matching connected. Let  $x \in V(G)$ ,  $M \in \mathcal{M}(G)$ , and  $e \in M$ . If  $x \in e$ ,  $e$  is a trivial strongly extendable path and we are done, so suppose  $x \notin e$ . For the sake of contradiction, without loss of generality, we assume that  $x$ ,  $e$ , and  $M$  are chosen to minimise  $\min_{v \in e} (\text{dist}_{\mathbf{cov}(G)}(x, v))$  and such that there is no strong  $x$ - $e$ - $M$ -extendable path in  $G$ , where  $v \in e$  is the witness of the distance between  $x$  and the endpoints of  $e$ . Consider a shortest  $x$ - $v$ -path in  $\mathbf{cov}(G)$ . Let  $u$  be the vertex along this path adjacent to  $v$ . Then clearly  $\text{dist}_{\mathbf{cov}(G)}(x, u) < \text{dist}_{\mathbf{cov}(G)}(x, v)$ . Furthermore, let  $e' \in M$  be the edge of  $M$  covering  $u$ . We get  $\min_{u' \in e'} (\text{dist}(x, u')) < \min_{v' \in e} (\text{dist}(x, v'))$  and thus, there exists a strong  $x$ - $e'$ - $M$ -extendable path  $P$ . Clearly,  $P$  does not contain a vertex of  $e$ , as then there would be a strong  $x$ - $e$ - $M$ -extendable path.

Now, since  $u$  and  $v$  are adjacent in the cover graph of  $G$ , there exists a perfect matching  $M' \in \mathcal{M}(G)$  with  $uv \in M'$ . Then, Lemma 2.6 certifies the existence of an  $M$ - $M'$ -alternating cycle  $C$  in  $\mathbf{cov}(G)$  which contains the edge  $uv$ . Moreover, since  $C$  is  $M$ - $M'$ -alternating and  $e, e' \in M$ , it contains both  $e$  and  $e'$  as well. Observe that if  $C \cap P$  consists only of the edge  $e'$  then we may use the union of one of the connected components of  $C - e'$  with  $P$  to obtain a strong  $x$ - $e$ - $M$ -extendable path. Therefore  $C \cap P$  have more than one edge in common. Moreover, if  $Q$  is a component of



$C \cap P$  then  $Q$  is an alternating path whose first and last edges belong to  $M$ , as otherwise there would be two edges of  $M$  sharing a common endpoint.

Observe that for all  $y \in V(C)$  there is a strong  $y$ - $e$ - $M$ -extendable path which is a subpath of  $C$ . Let  $y' \in V(P \cap C)$  be the vertex with the smallest distance to  $x$  along  $P$ . The edge  $e'' = y'y \in M$  belongs to  $C$  and thus, there exists a strong  $y$ - $e$ - $M$ -extendable path  $P' \subseteq C$ . By definition  $P'$  does not contain  $e''$  and thus the subpath  $P'' \subseteq P$  with endpoints  $x$  and  $y'$  together with the edge  $e''$  and the path  $P'$  form a strong  $x$ - $e$ - $M$ -extendable path. This completes the proof of our lemma.  $\square$

**Lemma 2.11** *For a graph  $G$  and any  $M \in \mathbf{cov}(G)$  let  $e, e' \in M$  and  $e' = xx'$ . There is a strong  $x$ - $e$ - $M$ -extendable path if and only if there is a strong  $x'$ - $e$ - $M$ -extendable path.*

PROOF. This follows directly from Theorem 2.10 and the definition of matching connectivity.  $\square$

**Corollary 2.12** *A graph  $G$  is matching connected if and only if for every perfect matching  $M \in \mathcal{M}(G)$  and pair of edges  $e, e' \in M$  there is a strong  $x$ - $e$ - $M$ -extendable path for some  $x \in e'$ .*

From Lemma 2.11, for any given pair of edges  $e, e' \in E(G) \cap M$  and any  $M \in \mathcal{M}(G)$ , with  $e = xx'$  there are two strongly extendable paths, one respecting  $x, e'$ , and  $M$  and the other one respecting  $x', e'$ , and  $M$ . These two paths might intersect each other many times, but whenever they do, they form cycles.

By combining this observation with Lemma 2.6 we obtain another characterisation of matching connectivity; this time in terms of intersecting  $M$ -alternating cycles.

Let  $G$  be a graph,  $x, y \in V(G)$ , and  $M \in \mathcal{M}(G)$ . A collection of  $M$ -alternating cycles  $C_1, \dots, C_k$  in  $M$ , is called an  $M$ -chain for  $x$  and  $y$ , if  $x \in V(C_1)$ ,  $y \in V(C_k)$  and  $V(C_i) \cap V(C_{i+1})$  is non-empty and  $M$ -central for all  $i \in \{1, \dots, k-1\}$ .

**Lemma 2.13** *A graph  $G$  is matching connected if and only if it consists of a single edge or for every perfect matching  $M \in \mathcal{M}(G)$  there is an  $M$ -chain for every pair of vertices  $x, y \in V(G)$ .*

PROOF. We first assume that  $G$  is matching connected. Without loss of generality we may also assume that  $G$  does not consist of a single edge. Let us fix an  $M \in \mathcal{M}(G)$  and  $x, y \in V(G)$ . Furthermore, let  $e \in M$  be the edge of  $M$  covering  $y$ . From Theorem 2.10 there is a strongly  $x$ - $e$ - $M$ -extendable path  $P$  in  $G$ . Let  $e_1, \dots, e_\ell$  be the edges of  $P$ , numbered in order of their appearance along  $P$  when traversing from  $x$  to  $y$ , that is,  $e_i \cap e_j \neq \emptyset$  if and only if  $|i - j| \leq 1$ . Note that  $\ell$  is odd and every edge with an even number is an edge of  $M$ .

We call a family  $\mathcal{C} = \{C_1, \dots, C_s\}$  of  $M$ -alternating cycles  $y$ -approaching with respect to the strongly  $x$ - $e$ - $M$ -extendable path  $P$  if the following hold:

1.  $x \in V(C_1)$ ,
2.  $V(C_i) \cap V(C_{i+1}) \neq \emptyset$  and  $M$ -central,
3.  $C_i \cap P \neq \emptyset$ , and
4.  $\text{dist}_P(C_i, y) < \text{dist}_P(C_{i+1}, y)$ ,  $i \in [s-1]$ .

We first prove that such a family exists. Towards this, let  $M_1$  be a perfect matching of  $G$  containing  $e_1$  with  $M \cap M_1 = \emptyset$ . Then, from Lemma 2.6 there is an  $M$ - $M_1$ -alternating cycle in  $G$ .

Since this cycle is  $M$ -alternating, it contains the edge  $e_2$  and the edge  $e' \in M$  that covers  $x$  in  $G$ . Therefore,  $C_1$  contains  $x$  and trivially  $\mathcal{C} = \{C_1\}$  forms such a family.

We claim that if  $\mathcal{C} = \{C_1, \dots, C_s\}$  is a maximal according to the number of members  $y$ -approaching family of  $M$ -alternating cycles with respect to the strongly  $x$ - $e$ - $M$ -extendable path  $P$  then  $y \in V(C_s)$ .

Let us assume to the contrary that  $\mathcal{C}$  is maximal but  $y \notin V(C_s)$ . Let then  $e_{q_s}$  denote the edge of  $C_s$  for which  $\text{dist}_P(C_s, y)$  is minimum. Then, clearly, as  $y \notin V(C_s)$ ,  $e_{q_s} \neq e$ . Observe that  $e_{q_s} \in M$ .

Let now  $M_2$  denote the perfect matching containing the edge  $e_{q_s+1}$  and notice that  $M_2 \neq M_1$  and  $M_2 \neq M$ . Let  $C'$  be the  $M$ - $M_2$ -alternating cycle that contains  $e_{q_s+1}$  (obtained from Lemma 2.6). Notice that since  $C'$  is  $M$ - $M_2$ -alternating it also contains the edge  $e_{q_s}$ , hence  $V(C_s) \cap V(C') \neq \emptyset$ . Moreover, since  $C_s$  and  $C'$  are  $M$ -alternating, for every vertex they have in common, they also share the corresponding edge of  $M$  covering this vertex. Hence  $V(C_s) \cap V(C')$  is  $M$ -central. Furthermore, as  $C'$  contains  $e_{q_s+1}$  it holds that  $C' \cap P \neq \emptyset$ . Finally, by construction,  $\text{dist}_P(C_s, y) < \text{dist}_P(C', y)$ .

Therefore, the family  $\mathcal{C}' = C_1, \dots, C_s, C'$  is also a  $y$ -approaching family of  $M$ -alternating cycles with respect to the strongly  $x$ - $e$ - $M$ -extendable path  $P$ . Moreover,  $\mathcal{C} < \mathcal{C}'$ , a contradiction to the maximality of  $\mathcal{C}$ . Hence,  $y \in V(C_s)$  and we obtain the  $M$ -chain.

If, on the other hand, for any matching  $M$  and every pair of vertices  $x, y \in V(G)$  we have an  $M$ -chain of cycles then from Lemma 2.5 we obtain that all edges of these cycles belong to some perfect matching of  $G$  and thus  $\mathbf{cov}(G)$  is connected.  $\square$

We have now gathered several descriptions of matching connectivity. We conclude this section with its main theorem: A collection of all the different characterisations of matching connectivity we stated before.

**Theorem 2.14** *Let  $G$  be a graph that is not an isolated edge. The following statements are equivalent.*

- (i)  $G$  is matching connected,
- (ii)  $\mathbf{cov}(G)$  is 2-connected,
- (iii) for all perfect matchings  $M \in \mathcal{M}(G)$ , every  $x \in V(G)$ , and every  $e \in M$  there is a strong  $x$ - $e$ - $M$ -extendable path,
- (iv) for all perfect matchings  $M \in \mathcal{M}(G)$  and every pair of edges  $e, e' \in M$  there is some  $x \in e'$  such that there is a strong  $x$ - $e$ - $M$ -extendable path, and
- (v) for all perfect matchings  $M \in \mathcal{M}(G)$  and every pair of vertices  $x, y \in V(G)$  there is an  $M$ -chain for  $x$  and  $y$ .

### 3 Matching $n$ -Connectivity

In this section we introduce the concept of matching separations for fixed perfect matchings as well as in general. This permits us to define higher orders of matching connectivity which will ultimately lead to a theorem analogous to Menger's Theorem on disjoint paths and  $n$ -connectivity.

In order to discuss about extendable paths connecting sets of vertices, we need to generalize the notion of single vertices and edges, both in relation to a fixed perfect matching  $M$ . For edges we already have the notion of  $M$ -central sets, but for vertices we need a better description.

Let  $G$  be a graph,  $M \in \mathcal{M}(G)$ , and  $X \subseteq V(G)$ . We call  $X$  *matching scattering* if  $G - X$  does not have a perfect matching. The set  $X$  is called  *$M$ -scattered* if  $|e \cap X| \leq 1$  for all  $e \in M$ . Moreover, it is called *strongly scattered* if it is  $M$ -scattered for all  $M \in \mathcal{M}(G)$ . Please note that an  $M$ -scattered or strongly scattered set is not necessarily matching scattering.

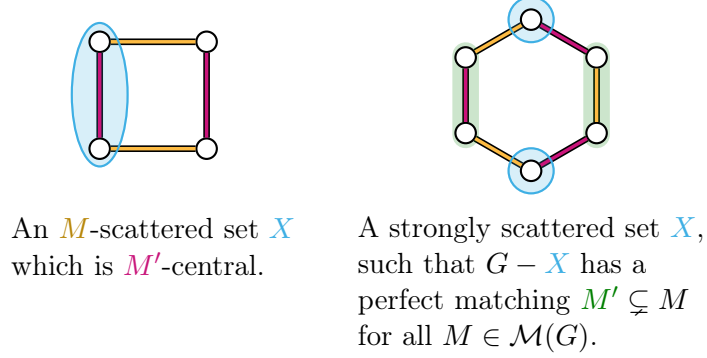


Figure 4: Examples of  $M$ -scattered and strongly scattered sets, that are not matching scattering.

Let  $X \subseteq V(G)$  be an  $M$ -scattered set and  $Y \subseteq V(G)$  be an  $M$ -central set. A weakly (respectively strongly) extendable path  $P$  is said to *respect*  $x$ ,  $Y$ , and  $M$ , if it is a weak (strong)  $x$ - $e$ - $M$ -extendable path with  $x \in X$ , and  $e \in M \cap E(G[Y])$ . We say that  $P$  is a *weak* (respectively *strong*)  $X$ - $Y$ - $M$ -extendable path.

Before we go on to the next definitions we first need to define the *order* of an  $M$ -central set  $S \subseteq V(G)$ , denoted by  $\text{ord}(S)$ , as  $\text{ord}(S) := \max_{M \in \mathcal{M}(G)} |E(G[S]) \cap M| = \nu(G[S])$ .

- Definition 3.1** (i) Let  $G$  be a graph and  $M \in \mathcal{M}(G)$ . Let  $X \subseteq V(G)$  be an  $M$ -scattered set and  $Y \subseteq V(G)$  be an  $M$ -central set. An  $M$ -central set  $S \subseteq V(G)$   *$M$ -separates*  $X$  and  $Y$ , if  $S$  contains a vertex of every strong  $X$ - $Y$ - $M$ -extendable path in  $G$ . We call  $S$  an  *$X$ - $Y$ - $M$ -separator*.
- (ii) Let  $G$  be a graph and  $S \subseteq V(G)$  be a strongly central set of  $G$ . Let then  $M \in \mathcal{M}(G)$  such that  $M \cap G[S] \in \mathcal{M}(G[S])$ ,  $X \subseteq V(G)$  be an  $M$ -scattered set, and  $Y \subseteq V(G)$  be an  $M$ -central set. If  $S$  is an  $X$ - $Y$ - $M$ -separator we call  $S$  an  *$X$ - $Y$ -matching separator*.
- (iii) Let  $G$  be a graph and  $M \in \mathcal{M}(G)$ . An  *$M$ -separator* of  $G$  is an  $M$ -central set  $S \subseteq V(G)$ , such that there are an  $M$ -scattered set  $X \subseteq V(G)$  and an  $M$ -central set  $Y \subseteq V(G)$  where  $S$  is an  $X$ - $Y$ - $M$ -separator,  $X \setminus S \neq \emptyset$ , and  $Y \setminus S \neq \emptyset$ .
- (iv) Let  $G$  be a graph and  $S \subseteq V(G)$  be a strongly central set of  $G$ . Let then  $M \in \mathcal{M}(G)$  such that  $M \cap G[S] \in \mathcal{M}(G[S])$ . The set  $S$  is a *matching separator* of  $G$  if there are sets  $X, Y \subseteq V(G)$ , where  $X$  is  $M$ -scattered and  $Y$  is  $M$ -central, such that  $S$  matching separates  $X$  and  $Y$ ,  $X \setminus S \neq \emptyset$ , and  $Y \setminus S \neq \emptyset$ .
- (v) Let  $G$  be a graph and  $M \in \mathcal{M}(G)$ . An  *$M$ -separation* in  $G$  is a pair  $(A, B)$  of  $M$ -central subgraphs of  $G$  such that there is a set  $F$  of edges with one endpoint in  $A$  and the other one in  $B$  with  $G - F = A \cup B$ ,  $F \cap M_F = \emptyset$  for all  $M_F \in \mathcal{M}(G - (V(A) \cap V(B)))$ ,  $A - B \neq \emptyset$ , and  $B - A \neq \emptyset$ .

The *order* of  $(A, B)$  is  $\text{ord}(V(A) \cap V(B))$ . We also write  $(A, S, B)$  for an  $M$ -separation with  $S = V(A) \cap V(B)$ .

- (vi) A *matching separation* in  $G$  is a pair  $(A, B)$  of strongly central subgraphs of  $G$ , with  $V(A) \cap V(B)$  also strongly central, such that there is a set  $F$  of edges with one endpoint in  $A$  and the other one in  $B$  with  $G - F = A \cup B$ ,  $F \cap M_F = \emptyset$  for all  $M_F \in \mathcal{M}(G - (V(A) \cap V(B)))$ ,  $A - B \neq \emptyset$ , and  $B - A \neq \emptyset$ .

The *order* of  $(A, B)$  is  $\text{ord}(V(A) \cap V(B))$ . We also write  $(A, S, B)$  for a matching separation with  $S = V(A) \cap V(B)$ .

It follows from the definition that if  $(A, S, B)$  is an  $M$ -separation for some  $M \in \mathcal{M}(G)$ , then  $S$  is  $M$ -central and an  $A$ - $B$ - $M$ -separator. Similar conclusions can be drawn for matching separations.

**Definition 3.2** A graph  $G$  is *matching  $n$ -connected* for some  $n \in \mathbb{N}$ , if  $|V(G)| \geq 2n + 4$  and  $G - X$  is matching connected for every strongly central set  $X \subseteq V(G)$  with  $\text{ord}(X) \leq n - 1$ .

**Lemma 3.3** *If  $G$  is a matching  $n$ -connected graph then it is also  $(n + 1)$ -connected.*

PROOF. Let  $G$  be a matching  $n$ -connected graph that is not  $(n + 1)$ -connected. Then we can find a separator  $S \subseteq V(G)$  with  $|S| \leq n$ .

We begin by showing that  $S$  is an independent set of  $\mathbf{cov}(G)$ . Suppose otherwise that  $S$  is not independent in  $\mathbf{cov}(G)$ . Then there is some perfect matching  $M \in \mathcal{M}(G)$  and edge  $e' \in M$  such that  $e' \subseteq S$ . Consider

$$S_M := \bigcup_{e \in M: e \cap S \neq \emptyset} e,$$

as the smallest  $M$ -central superset of  $S$ . Since  $e' \subseteq S$ , we get  $\text{ord}(S_M) \leq n - 1 < n$ . Notice that if  $S_M$  is a matching separator of  $G$  then it is not matching  $n$ -connected, a contradiction to the hypothesis. Thus  $S_M$  is not a matching separator of  $G$ . Let  $X$  and  $Y$  denote a partition of the connected components of  $G \setminus S$  where  $X \neq \emptyset$  and  $Y \neq \emptyset$ . Let

$$X_M = \{v \in X : v \cap e \neq \emptyset \text{ for every } e \in M \text{ with } e \subseteq S_M\}$$

and

$$Y_M = \{v \in Y : v \cap e \neq \emptyset \text{ for every } e \in M \text{ with } e \subseteq S_M\}.$$

Recall that, since  $G$  is matching  $n$ -connected,  $|V(G)| \geq 2n + 4$ . Moreover, by construction  $|S_M| \leq 2n$ . This implies that  $X \setminus X_M \neq \emptyset$  or  $Y \setminus Y_M \neq \emptyset$ . In particular, we prove that at least one of the graphs  $X \setminus X_M$  and  $Y \setminus Y_M$  is empty. Let us assume that both of the graphs  $X \setminus X_M$  and  $Y \setminus Y_M$  are not empty. Observe that  $X \setminus X_M$  and  $Y \setminus Y_M$  are strongly central. It follows that we can construct a set  $P$  in  $X \setminus X_M$  that is  $M$ -scattered by picking exactly one endpoint of every edge of  $M$  in  $X \setminus X_M$ . It follows then that  $S_M$  matching separates  $P$  and  $Y \setminus Y_M$ . This is a contradiction to the fact that  $\text{ord}(S_M) \leq n - 1$  and the assumption that  $G$  is matching  $n$ -connected. Hence, exactly one of the graphs  $X \setminus X_M$  and  $Y \setminus Y_M$  is empty. Without loss of generality, let us assume that  $X \setminus X_M$  is empty. Notice that if  $X_M = \emptyset$  then  $X = X \setminus X_M = \emptyset$ , a contradiction to the choice of  $X$ . Thus, there exists at least one vertex  $x$  of  $X$  in  $X_M$ . Let  $e_x \in M$  such that  $x \in e_x$  and  $e_x \subseteq S_M$ . That is, let  $e_x$  be the edge of  $M$  that is entirely contained in  $S_M$  and covers  $x$ . Let  $S_M^x = S_M \setminus e_x$ . Observe that  $S_M^x$  matching separates the sets  $\{x\}$  and  $Y \setminus Y_M$ . Since  $Y \setminus Y_M \neq \emptyset$  and the set  $Y \setminus Y_M$

is also strongly central, the set  $S_M^x$  is a matching separator with  $\text{ord}(S_M^x) = \text{ord}(S_M) - 1 < n$ , a contradiction. Hence  $S$  is an independent set of  $\mathbf{cov}(G)$ .

Consider any perfect matching  $M \in \mathcal{M}(G)$  and let  $S_M, X, X_M, Y$ , and  $Y_M$  be defined as above. Suppose first that  $X \setminus X_M \neq \emptyset$  and  $Y \setminus Y_M \neq \emptyset$  and let  $u \in X \setminus X_M$  and  $v \in Y \setminus Y_M$ . Let  $e \subseteq S_M$  be any edge with  $e = xy \in M$ . Since  $S$  is independent in  $\mathbf{cov}(G)$  it follows that only one endpoints of  $e$ , say  $x$  belongs to  $S$  and that  $\text{ord}(S_M) = n$ . Thus  $\text{ord}(S_M \setminus \{x, y\}) = n - 1$  and  $G - (S_M \setminus \{x, y\})$  is matching connected. By Theorem 2.14, (ii) since  $G - (S_M \setminus \{x, y\})$  is matching connected it is also 2-connected. By Menger's theorem, there exist two disjoint paths between  $u$  and  $v$ . Thus there exists a path between  $u$  and  $v$  avoiding  $S_M \setminus \{x, y\}$  and  $x$ . Since  $S \subseteq S_M \setminus \{y\}$ , there exists a path between  $u$  and  $v$  avoiding  $S$ . This is a contradiction since  $u$  and  $v$  belong to different connected components of  $G \setminus S$ .

Suppose then that at least one of  $X \setminus X_M$  and  $Y \setminus Y_M$ , say  $X \setminus X_M$ , is empty. Recall that, since  $G$  is matching  $n$ -connected,  $|V(G)| \geq 2n + 4$ . Moreover, by construction and the fact that  $S$  is independent in  $\mathbf{cov}(G)$  we obtain that  $|S_M| = 2n$ . This implies that  $Y \setminus Y_M \neq \emptyset$ . Let  $p \in Y \setminus Y_M$ . Observe that if  $X_M = \emptyset$  then  $X = \emptyset$ , a contradiction since  $X$  and  $Y$  is a partition of the connected components of  $G \setminus S$ . Let then  $e = xy \subseteq S_M$  be an edge of  $S_M$  that has an endpoint in  $X_M$ . Without loss of generality let  $x$  be that endpoint. As before  $\text{ord}(S_M \setminus \{x, y\}) = n - 1$  and hence the graph  $G \setminus (S_M \setminus \{x, y\})$  is matching connected. By Theorem 2.14, (ii) since  $G - (S_M \setminus \{x, y\})$  is matching connected it is also 2-connected. By Menger's theorem, there exist two disjoint paths between  $x$  and  $p$ . Thus there exists a path between  $x$  and  $p$  avoiding  $S_M \setminus \{x, y\}$  and  $y$ . Since  $S \subseteq S_M \setminus \{x\}$ , there exists a path between  $x$  and  $p$  avoiding  $S$ . This is a contradiction since  $x$  and  $p$  belong to different connected components of  $G \setminus S$ .  $\square$

**Lemma 3.4** *Let  $G$  be a graph and  $v \in V(G)$  be a vertex of  $G$  of degree  $n + 1$ ,  $n \geq 1$ . If  $G$  is matching  $n$ -connected then  $N(v)$  is an independent set of  $\mathbf{cov}(G)$ .*

PROOF. Let  $N(v) = \{x_1, \dots, x_{n+1}\}$  and suppose that two of those vertices, say  $x_1$  and  $x_2$ , are adjacent in the cover graph of  $G$ . Then there exists a perfect matching  $M$  containing the edge  $x_1x_2$ . Moreover,  $M$  covers  $v$  and all other neighbours of  $v$ . Without loss of generality, let  $vx_{n+1} \in M$ . We denote by  $y_3, \dots, y_n$  the vertices for which  $x_iy_i \in M$  for all  $i \in \{3, \dots, n\}$ . Note that it's possible that some of the  $y_i$ s belong to  $N(v)$ .

It follows that  $S := \{x_1, \dots, x_n, y_3, \dots, y_n\}$  is  $M$ -central and  $\text{ord}(S) \leq n - 1$ . Moreover, since  $G$  is matching  $n$ -connected, it holds that  $|V(G)| \geq 2n + 4$ . Therefore,  $G - S$  contains at least six vertices, since  $|S| \leq 2n - 2$ . Furthermore,  $v$  is left with exactly one neighbour, the vertex  $x_{n+1}$ , in  $G - S$ . Therefore, the graph  $G - S$  is not 2-connected and thus from Theorem 2.14, (ii) it is not matching connected. Thus,  $S$  is a matching separator of order at most  $n - 1$ , a contradiction.  $\square$

We now have acquired enough tools and knowledge on matching connectivity to aim for the first main result of this paper: A Menger-type characterization of matching connectivity. As before we start out with a result for a fixed perfect matching  $M$ , which we then extend towards general matching connectivity.

**Theorem 3.5** *For every graph  $G$ ,  $M \in \mathcal{M}(G)$ ,  $M$ -scattered set  $X \subseteq V(G)$  and  $M$ -central  $Y \subseteq V(G)$  the minimum order of an  $M$ -central set separating  $X$  from  $Y$  is equal to the maximum number of disjoint strong  $X$ - $Y$ - $M$ -extendable paths.*

PROOF. We show by induction on  $|V(G)| + |E(G)|$  that if the minimum  $G$  contains  $n$  disjoint strong  $X$ - $Y$ - $M$ -extendable paths. Notice that the base of the induction is the graph that consists of a single edge and the lemma holds trivially. Let now  $G$  be a graph that does not consist of a single edge.

Consider first the case where  $X \cap Y \neq \emptyset$ . Let then  $v \in X \cap Y$ . Clearly  $v$  is covered by some edge  $vu \in M$  and let  $G' = G - v - u$ . Then  $|V(G')| + |E(G')| < |V(G)| + |E(G)|$ . Let  $X' = X \setminus \{v, u\}$  and  $Y' = Y \setminus \{v, u\}$ . It follows that the minimum order of an  $X'$ - $Y'$ - $M$ -separator in  $G'$  is at least  $n - 1$ . Hence by induction there are at least  $n - 1$  disjoint  $X'$ - $Y'$ - $M$ -extendable paths in  $G - u - v$ . These paths are also disjoint strong  $X$ - $Y$ - $M$ -extendable paths in  $G$ . Moreover, since  $uv$  is a trivial extendable path in  $G$ , there exist  $n$  disjoint strong  $X$ - $Y$ - $M$ -extendable paths in  $G$ .

Therefore we now assume that  $X \cap Y = \emptyset$ . Suppose that  $S$  is an  $X$ - $Y$ - $M$ -separator with  $X \not\subseteq S$ ,  $Y \neq S$  and order exactly  $n$ . Let  $C_X \subseteq G - S$  be the graph consisting of all components of  $G - S$  that contain at least one vertex of  $X$  and  $C_Y$  be the graph consisting of all components of  $G - S$  that contain at least one vertex of  $Y$ . Notice that  $|X| \geq n$  and  $\text{ord}(Y) \geq n$  as otherwise there would be an  $X$ - $Y$ - $M$ -separator of order strictly less than  $n$ . Since  $X \subsetneq S$ ,  $C_X \neq \emptyset$ . Moreover, since  $Y \neq S$ ,  $C_Y \neq \emptyset$ . Furthermore, since  $S$  is an  $X$ - $Y$ - $M$  separator, the sets  $C_X$  and  $C_Y$  are disjoint. Consider then the graphs  $G_X$  and  $G_Y$ , which are the subgraphs of  $G$  induced by the vertex sets  $V(C_X) \cup S$  and  $V(C_Y) \cup S$ , respectively. Notice that every  $X$ - $S$ - $M$ -separator of  $G_X$  is also an  $X$ - $Y$ - $M$ -separator of  $G$  and thus, every  $X$ - $S$ - $M$ -separator of  $G_X$  has order at least  $n$ . Moreover, let  $S' \subseteq S$  be an  $M$ -scattered set of  $G_Y$ , then every  $S'$ - $Y$ - $M$ -separator of  $G_Y$  is also an  $X$ - $Y$ - $M$ -separator of  $G$ . Hence every  $S'$ - $Y$ - $M$ -separator of  $G_Y$ , where  $S' \subseteq S$  is an  $M$ -scattered set, has order at least  $n$ .

Since  $G_X \subsetneq G$ , the induction gives us the existence of  $n$  disjoint strong  $X$ - $S$ - $M$ -extendable paths in  $G_X$ . Let  $\mathcal{P}_X$  be a family of  $n$  such paths. Since  $S$  is  $M$ -central and of order exactly  $n$  there are exactly  $n$  edges of  $M$  in  $G[S]$ . Thus for every  $P_i^X \in \mathcal{P}_X$ ,  $1 \leq i \leq n$ , there is a unique edge  $s_i x_i \in E(G[S]) \cap M$  such that  $s_i$  is the endpoint of  $P_i^X$ . If  $P_i^X$  is trivial, choose  $s_i$  to be the unique endpoint of  $P_i^X$  that is not contained in  $X$ . This uniquely induces a partition of  $S$  into two  $M$ -scattered sets  $X' := \{x_1, \dots, x_n\}$  and  $S \setminus X' = \{s_1, \dots, s_n\}$ , where  $S \setminus X'$  is the set of endpoints of the paths in  $\mathcal{P}_X$  (see Figure 5).

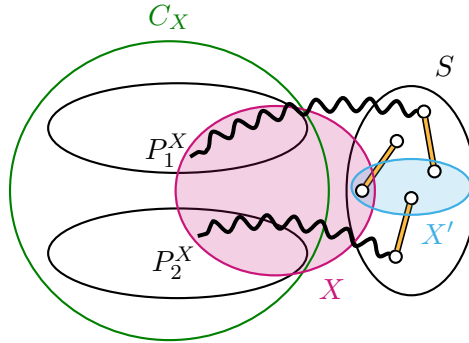
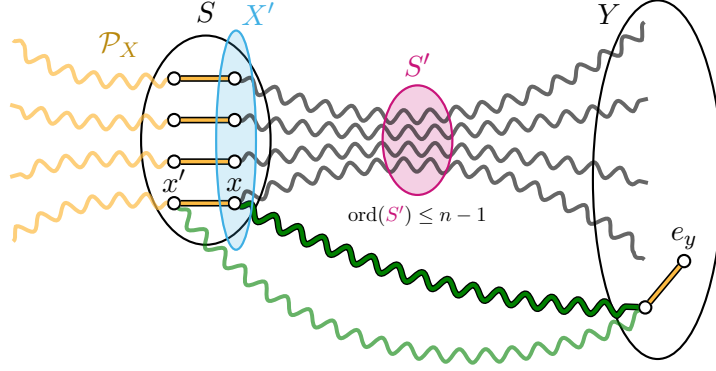


Figure 5: The construction of the  $M$ -scattered set  $X'$  in  $G_X$ .

Now consider  $G_Y$ . Recall that there is no  $X'$ - $Y$ - $M$ -separator of order strictly less than  $n$  in  $G_Y$ . Hence, by induction, we can find  $n$  disjoint  $X'$ - $Y$ - $M$ -extendable paths in  $G_Y$ . Let  $\mathcal{P}_Y$  be a family of such paths. For every  $P_i^Y \in \mathcal{P}_Y$  let  $x_i$  be its endpoint in  $X'$ ,  $1 \leq i \leq n$ . If  $P_i^Y$  is trivial, clearly



□

Figure 6: The  $X'$ - $Y$ - $M$ -separator  $S'$  of order at most  $n - 1$  and the  $x'$ - $e_y$ - $M$ -extendable path contradicting the existence of  $S'$ .

$V(P_i^Y) = \{s_i, x_i\} \subseteq S \cap Y$ . By definition no non-trivial path  $P_i^Y$  contains a vertex of  $S \setminus X'$ .

Now back to  $G$ . Every non-trivial strongly extendable path  $P_i^X$  has  $s_i$  as one of its endpoints, and if  $P_i^Y$  is non-trivial as well, it has  $x_i$  as its endpoint. The edge  $s_i x_i \in M$  connects the two paths and thus  $P_i^X \cup \{s_i x_i\} \cup P_i^Y$  is an  $X$ - $Y$ - $M$ -extendable path  $P_i$ . If  $P_i^Y$  is trivial, then  $\{s_i, x_i\} \subseteq Y$  and thus  $P_i := P_i^X$  is already an  $X$ - $Y$ - $M$ -extendable path. So suppose  $P_i^X$  is trivial. If  $P_i^Y$  is so as well, we are done, so suppose it is not. Then, by construction of  $X'$ , there is a unique vertex  $x_i \in X \cap X' \cap V(P_i^X)$ , which is an endpoint of  $P_i^Y$ . Hence  $P_i := P_i^Y$  is already an  $X$ - $Y$ - $M$ -extendable path. Thus we have found  $n$  disjoint strongly  $X$ - $Y$ - $M$ -extendable paths.

Finally, let us assume that  $S$  is an  $X$ - $Y$ - $M$ -separator of order exactly  $n$  with either  $X \subseteq S$  or  $Y = S$ . Observe here that if both  $X \subseteq S$  and  $Y = S$  then  $X \cap Y \neq \emptyset$  and we have already proved this case. So either  $X \subseteq S$  and  $Y \neq S$  or  $X \not\subseteq S$  and  $Y = S$ . Let  $P$  be any strongly  $X$ - $Y$ - $M$ -extendable path in  $G$ . Let us assume that  $P$  consists of a single edge  $e$ , where  $e = xy \in M$  with  $x \in X$  and  $y \in Y$ . Then since  $Y$  is  $M$ -central,  $\{x, y\} \subseteq Y$  and thus  $X \cap Y \neq \emptyset$ , a contradiction to the assumption that  $X \cap Y = \emptyset$ . Thus,  $P$  consists of at least 2 edges. Therefore, there is an edge  $xy$  of  $P$  such that  $xy \notin M$ ,  $x \notin Y$  and  $y \notin X$ . Notice that  $M \in \mathcal{M}(G - xy)$  and let  $x', y' \in V(G)$  be the two distinct vertices such that  $xx', yy' \in M$ . Let  $Z$  be a minimum  $X$ - $Y$ - $M$ -separator in  $G - xy$ ,  $Z_X := Z \cup \{x, x'\}$ , and  $Z_Y := Z \cup \{y, y'\}$ . Clearly both  $Z_X$  and  $Z_Y$  are  $X$ - $Y$ - $M$ -separators in  $G$  and therefore  $\text{ord}(Z_X) \geq n$  and  $\text{ord}(Z_Y) \geq n$ . Moreover,  $Z_X \neq Z_Y$  and the strongly central sets  $Z_X$  and  $Z_Y$  have the same order.

Suppose first that both equalities,  $\text{ord}(Z_X) = n$  and  $\text{ord}(Z_Y) = n$ , holds. Observe that if  $X \not\subseteq Z_X$  and  $Y \neq Z_X$ , we already have  $n$  disjoint strongly  $X$ - $Y$ - $M$ -extendable paths by a previous case. Hence either  $X \subseteq Z_X$  or  $Y = Z_X$ . Since  $x \in Z_X$  and  $x \notin Y$ , it follows that  $X \subseteq Z_X$ . In particular,  $Z_X = X^*$ , where  $X^* := \{v \in V(G) \mid v \in e, \text{ with } e \in M \text{ and } e \cap X \neq \emptyset\}$ . Moreover, if  $X \not\subseteq Z_Y$  and  $Y \neq Z_Y$ , we already have  $n$  disjoint strongly  $X$ - $Y$ - $M$ -extendable paths by a previous case. Hence either  $X \subseteq Z_Y$  or  $Y = Z_Y$ . However, if  $X \subseteq Z_Y$   $X^* = Z_Y \neq \emptyset$  and then  $Z_X = Z_Y$ , a contradiction. Notice now that if  $Y = Z_Y$ ,  $Z = Z_X \cap Z_Y = X^* \cap Y$ . Since  $|Z| \geq n - 1 \geq 1$ , it follows that  $X \cap Y \neq \emptyset$ , a contradiction to the hypothesis that  $X \cap Y = \emptyset$ . Therefore, we obtain that  $\text{ord}(Z_X) > n$  and  $\text{ord}(Z_Y) > n$ . Hence  $\text{ord}(Z) \geq n$ . Since  $Z$  is an  $X$ - $Y$ - $M$ -separator of minimum order in  $G - xy$  and  $\text{ord}(Z) \geq n$ , by the induction hypothesis there exist  $n$  disjoint strong



$X$ - $Y$ - $M$ -extendable paths in  $G - xy$ . The fact that  $G - xy \subsetneq G$  concludes the proof.

In order to build the road to a more general Menger-type result, independent of  $M$ , we need some more refinements of the above result.

Let  $G$  be a graph,  $M \in \mathcal{M}(G)$ ,  $x \in V(G)$  and  $Y \subseteq V(G)$   $M$ -central. Then a family of strongly  $x$ - $Y$ - $M$ -extendable paths are called an  $x$ - $Y$ - $M$ -extendable fan if any two of the paths have only  $x$  in common.

**Corollary 3.6** *For every graph  $G$ ,  $M \in \mathcal{M}(G)$ ,  $x \in V(G)$ , and  $M$ -central set  $Y \subseteq V(G)$ , the minimum order of an  $M$ -central set  $M$ -separating  $x$  from  $Y$  is equal to the maximum number of strong  $x$ - $Y$ - $M$ -extendable paths that form an  $x$ - $Y$ - $M$ -extendable fan.*

PROOF. Let  $xx' \in M$  be the edge of  $M$  covering  $x$ . Let also  $X' := N(x) \setminus \{x'\}$ . While there exists an edge  $e' \subseteq X'$ , with  $e' \in M$ , chose a vertex  $z \in e'$  and remove it from  $X'$ . Observe that the set  $X$  obtained in this way is  $M$ -scattered. Notice that the number of strong  $x$ - $Y$ - $M$ -extendable paths that form an  $x$ - $Y$ - $M$ -extendable fan is equal to the number of disjoint strong  $X$ - $Y$ - $M$ -extendable paths. The corollary then follows by Theorem 3.5.  $\square$

For an even more refined version of Theorem 3.5 regarding the number of strongly extendable paths for a single vertex  $x$  and a given edge  $e$  of some perfect matching  $M$  we need weaker version of disjoint paths. So we call two strongly  $x$ - $e$ - $M$ -extendable paths  $P_1, P_2$  independent, if  $V(P_1) \setminus (\{x\} \cup e)$  is disjoint from  $V(P_2) \setminus (\{x\} \cup e)$ .

**Corollary 3.7** *Let  $G$  be a graph,  $M \in \mathcal{M}(G)$ ,  $x \in V(G)$  and  $e \in M$ . If  $x \notin e$  and  $e \cap N(x) = \emptyset$ , then the minimum order of an  $M$ -central set  $M$ -separating  $x$  from  $e$  is equal to the maximum number of independent  $x$ - $e$ - $M$ -extendable paths.*

PROOF. Let  $e = y_1y_2$  and  $xx'$  be the edge of  $M$  covering  $x$  with  $xx' \neq e$ . Let also  $X' := N(x) \setminus \{x'\}$ . While there exists an edge  $e' \subseteq X'$ , with  $e' \in M$ , chose a vertex  $z \in e'$  and remove it from  $X'$ . Observe that the set  $X$  obtained in this way is  $M$ -scattered. We define the set  $Y := \{y_1, y_2\}$ . Then apply Theorem 3.5 with  $X, Y$ , and  $M$ . At last extend the resulting disjoint  $X$ - $Y$ - $M$ -extendable paths accordingly to acquire the desired  $x$ - $e$ - $M$ -extendable paths.  $\square$

We are now ready to state and proof the main result of this section: Characterising matching  $n$ -connectivity by the number of independent  $x$ - $e$ - $M$ -extendable paths. This concludes this section.

**Theorem 3.8** *A graph  $G$  is matching  $n$ -connected if and only if for all perfect matchings  $M \in \mathcal{M}(G)$ , all vertices  $x \in V(G)$ , and all  $e \in M$  with  $x \notin e$ , there are  $n$  independent  $x$ - $e$ - $M$ -extendable paths in  $G$ .*

PROOF. Suppose there are  $n$  independent  $x$ - $e$ - $M$ -extendable paths in  $G$  for every  $x \in V(G)$ ,  $M \in \mathcal{M}(G)$ , and  $e \in M$ . Then clearly any strongly central set separating  $x$  from  $e$  is  $M$ -central and it  $M$ -separates  $x$  from  $e$ . Hence any such separator is at least of order  $n$  by Corollary 3.7.

On the other hand, suppose  $G$  is matching  $n$ -connected. Thus, any matching separator of  $G$  is at least of order  $n$ . Suppose there is some  $M \in \mathcal{M}(G)$ , a vertex  $x \in V(G)$ , and an edge  $e \in M$ , where  $x \notin e$ , such that there are no  $n$  independent  $x$ - $e$ - $M$ -extendable paths in  $G$ . Then, by Corollary 3.7, since  $x \notin e$ ,  $x$  is adjacent to one of the endpoints of  $e$ .

Say  $y \in e \cap N(x)$  and let  $G_1 := G - xy$ . Then  $M$  is clearly still a perfect matching of  $G_1$  and there are at most  $n - 2$  independent  $x$ - $e$ - $M$ -extendable paths in  $G_1$ . From [Corollary 3.7](#), we can  $M$ -separate, and therefore matching separate,  $x$  from  $e$  by some  $M$ -central set  $S$  of order at most  $n - 2$ . As  $|V(G)| \geq 2n + 4$  there is at least one other vertex  $v \notin S \cup \{x, x'\} \cup e$ . Moreover,  $S$   $M$ -separates  $v$  either from  $xx'$ , or from  $e$ . Without loss of generality, let  $S$   $M$ -separate  $v$  from  $xx'$ . Then  $S \cup e$  is an  $M$ -central set of order  $n - 1$   $M$ -separating, and therefore matching separating,  $v$  from  $xx'$ . Hence  $G$  cannot be matching  $n$ -connected, a contradiction.  $\square$

Concluding this section, we show that we can use [Lemma 2.8](#) together with Menger's Theorem for directed graphs and [Theorem 3.8](#) to obtain the following result on the strong connectivity of directed graphs.

**Theorem 3.9** *A bipartite graph  $G = (U \cup W, E)$  is matching  $n$ -connected for  $n \geq 1$  if and only if  $\mathcal{D}(G, M)$  is strongly  $n$ -connected for all  $M \in \mathcal{M}(G)$ .*

PROOF. Note first that  $G$  is matching  $n$ -connected if and only if  $\mathbf{cov}(G)$  is matching  $n$ -connected. Hence we can assume  $G$  to be matching covered and so all weakly extendable paths are also strongly extendable paths.

Suppose first that  $G$  is matching  $n$ -connected and pick any perfect matching  $M$  together with two distinct edges  $e_1, e_2 \in M$ . For  $i \in \{1, 2\}$  let  $u_i \in e_i \cap U$  and  $w_i \in e_i \cap W$ . As  $G$  is matching  $n$ -connected, [Theorem 3.8](#) asserts the existence of  $n$  internally vertex disjoint strongly  $u_1$ - $e_2$ - $M$ -extendable paths  $P_1, \dots, P_n$  as well as  $n$  internally vertex disjoint strongly  $u_2$ - $e_1$ - $M$ -extendable paths  $P'_1, \dots, P'_n$ . Since these paths are also weakly extendable from [Lemma 2.8](#) we obtain the existence of  $n$  vertex disjoint directed paths from  $v_1$  to  $v_2$  and  $n$  vertex disjoint directed paths from  $v_2$  to  $v_1$  in the  $M$ -direction  $\mathcal{D}(G, M)$  of  $G$ . Hence by Menger's Theorem,  $\mathcal{D}(G, M)$  is strongly  $n$ -connected.

For the reverse direction, let us assume that  $\mathcal{D}(G, M)$  is strongly  $n$ -connected. Then for every two vertices  $v, u \in V(\mathcal{D}(G, M))$  there exist  $n$  internally disjoint directed paths  $P_1, P_2, \dots, P_n$  from  $u$  to  $v$  and  $n$  internally disjoint directed paths  $Q_1, Q_2, \dots, Q_n$  from  $v$  to  $u$ . Let  $e_v = x_v y_v$  be the edge of  $M$  in  $G$  corresponding to  $v$  and  $e_u = x_u y_u$  be the edge of  $M$  in  $G$  corresponding to  $u$ , with  $x_v, x_u \in U$  and  $y_v, y_u \in W$ . From [Lemma 2.8](#) there exist  $n$   $x_v$ - $e_u$ - $M$ -weakly extendable paths  $P_1^G, P_2^G, \dots, P_n^G$  in  $G$  and  $n$   $x_u$ - $e_v$ - $M$ -weakly extendable paths  $Q_1^G, Q_2^G, \dots, Q_n^G$  in  $G$ . Observe now that, in order to show that  $G$  is matching  $n$ -connected, from [Theorem 3.8](#), it is enough to show that the above paths are also strongly extendable. To see this recall first that, from [Lemma 2.8](#), the edges of the paths  $P_i^G$  and  $Q_i^G$ ,  $i \in [n]$ , are mirrored by the edges of the paths  $P_i$  and  $Q_i$ ,  $i \in [n]$ . Observe now that each graph  $H_i = P_i \cup Q_i$ ,  $i \in [n]$ , is the union of directed cycles in  $\mathcal{D}(G, M)$ . Therefore, from [Lemma 2.9](#), the edges of the paths  $P_i^G$  and  $Q_i^G$ ,  $i \in [n]$ , that do not belong to  $M$ , belong to  $E(\mathbf{cov}(G)) \setminus M$ . Hence, the paths  $P_i^G$  and  $Q_i^G$ ,  $i \in [n]$  are strongly extendable. This completes the proof of the theorem.  $\square$

## 4 $n$ -Extendability

We are now ready to establish a first connection between matching connectivity and the canonical matching theory. To do this, we are going to establish a close relation between the (matching) extendability and the matching connectivity of a graph. Recall that a connected graph is  $n$ -extendable

for some  $n \in \mathbb{N}$ , if every matching of order  $n$  in  $G$  is contained in some perfect matching of the graph. So in particular, a graph is 1-extendable, if it is matching covered.

This section consists of two parts. First we present some basic results on  $n$ -extendability and connectivity, which will lead to the first major result of this section, stating that any  $n$ -extendable graph is matching  $n$ -connected. Then, in a series of lemmas, we will make our way towards a description of matching  $(n + 1)$ -connected but not  $n$ -extendable graphs.

Between these two parts we will take a brief detour to bipartite graphs, a class of graphs on which  $n$ -extendability and matching  $n$ -connectivity are equivalent.

To start out we will have a look at a first clue, besides [Theorem 1.4](#), on how extendability might be connected to matching connectivity. This is given by a classical result of Plummer.

**Lemma 4.1 (Plummer [6])** *Let  $G$  be a graph with  $|V(G)| \geq 2n + 2$  for some  $n \geq 1$ . If  $G$  is  $n$ -extendable, it is  $(n + 1)$ -connected.*

So similar to [Lemma 3.3](#),  $n$ -extendability implies high ordinary connectivity in a graph. Moreover, a 1-extendable graph is 2-connected. Furthermore, since a 1-extendable graph is matching covered, its cover graph is also 2-connected. Hence, we obtain the following.

**Corollary 4.2** *A graph  $G$  is matching connected if and only if  $\mathbf{cov}(G)$  is 1-extendable.*

[Corollary 4.2](#) is an illustrative example of the type of results we can expect in this section. While matching connectivity does not regard edges not contained in any perfect matching, extendability is a property of matching covered graphs exclusively. Hence here we are forced to always consider the cover graph of a given graph.

The next theorem shows that high extendability even implies high matching connectivity. So at least one direction of [Corollary 4.2](#) is preserved for  $n \geq 2$ . In order to do this, we first need to gain more insight into the behaviour of the cover graph, if it is  $n$ -extendable.

**Lemma 4.3** *Let  $G$  be a matching connected graph and  $n \geq 1$ . Then  $\mathbf{cov}(G)$  is  $n$ -extendable if and only if  $\mathbf{cov}(G - S) = \mathbf{cov}(G) - S$  for every strongly central set  $S \subseteq V(G)$  of order at most  $n - 1$ .*

PROOF. Suppose first that  $\mathbf{cov}(G)$  is  $n$ -extendable and let  $S \subseteq V(G)$  be a strongly central set with  $\text{ord}(S) \leq n - 1$ . Since  $V(G) = V(\mathbf{cov}(G))$ , it trivially holds that  $V(\mathbf{cov}(G - S)) = V(\mathbf{cov}(G) - S)$  so only the edge sets of the graphs  $\mathbf{cov}(G - S)$  and  $\mathbf{cov}(G) - S$  concern us from now on. We first prove that  $E(\mathbf{cov}(G - S)) \subseteq E(\mathbf{cov}(G) - S)$ . Let  $e \in E(\mathbf{cov}(G - S))$ . Then  $e \in G - S$  and there is  $M \in \mathcal{M}(G - S)$  such that  $e \in M$ . Since  $e \cap S = \emptyset$ , observe that to show  $e \in E(\mathbf{cov}(G) - S)$  it is enough to show that  $e \in E(\mathbf{cov}(G))$ . Since  $S$  is a strongly central set there exists a matching  $M'$  which is a matching of  $G[S]$  of order at most  $n - 1$ . Then  $M' \cup \{e\}$  is a matching of  $G$  of order at most  $n$ . Since  $\mathbf{cov}(G)$  is  $n$ -extendable there exists a matching  $M'' \in \mathcal{M}(G)$  of  $\mathbf{cov}(G)$  such that  $M' \cup \{e\} \subseteq M''$ . Thus, there exists  $M'' \in \mathcal{M}(G)$  with  $e \in M''$ . This implies that  $e \in \mathbf{cov}(G)$  and hence  $E(\mathbf{cov}(G - S)) \subseteq E(\mathbf{cov}(G) - S)$ .

Suppose now that there is some  $e \in E(\mathbf{cov}(G) - S)$ . Observe first that  $e \in \mathbf{cov}(G)$ . Moreover, since  $G[S]$  is a strongly central set of  $G$  there exists a matching  $M$  such that  $M \in \mathcal{M}(G[S])$  and hence  $M \subseteq E(\mathbf{cov}(G))$ . Since  $\text{ord}(S) \leq n - 1$ ,  $M \cup \{e\}$  is a matching of  $\mathbf{cov}(G)$  of order at most  $n$ . Therefore, from the  $n$ -extendability of  $\mathbf{cov}(G)$  there is a matching  $M'$  of  $\mathbf{cov}(G)$  containing  $M \cup \{e\}$ . Notice that  $M'$  is a matching of  $G$  and  $M'' = M' \setminus M$  is a matching of  $G - S$  with  $e \in M''$ . Hence,  $e \in E(\mathbf{cov}(G - S))$ .

The reverse direction is shown by induction on  $n$ . For  $n = 1$  notice that for every edge in  $\mathbf{cov}(G)$  there is a matching that contains it and hence,  $\mathbf{cov}(G)$  is 1-extendable. Thus, we can move on to  $n \geq 2$ . Assume  $\mathbf{cov}(G - S) = \mathbf{cov}(G) - S$  for every strongly central set  $S \subseteq V(G)$  of order at most  $n - 1$ . Then the equality also holds for every  $S$  of order at most  $n - 2$ . Hence, by induction,  $\mathbf{cov}(G)$  is  $(n - 1)$ -extendable. Let then  $M$  be any matching of order  $n - 1$  in  $\mathbf{cov}(G)$ . Since  $\mathbf{cov}(G)$  is  $(n - 1)$ -extendable, the set  $S_M$  of all endpoints of the edges of  $M$  is strongly central and thus  $\mathbf{cov}(G) - S_M = \mathbf{cov}(G - S_M)$ . This implies that every edge of  $\mathbf{cov}(G - S_M)$  is contained in a perfect matching of  $G - S_M$  and since  $S_M$  is a strongly central set, every edge of  $G$  is contained in a perfect matching that also contains  $M$  as a subset. Therefore, every matching of order  $n$  with  $M$  as a subset is contained in a perfect matching of  $G$ . Thus  $\mathbf{cov}(G)$  is  $n$ -extendable.  $\square$

**Theorem 4.4** *Let  $G$  be a graph with  $|V(G)| \geq 2n + 4$  for some  $n \geq 1$ . If  $\mathbf{cov}(G)$  is  $n$ -extendable,  $G$  is matching  $n$ -connected.*

PROOF. We prove the claim by induction over  $n$ . The case  $n = 1$  is already handled by Corollary 4.2. Let  $n = 2$  and suppose  $G$  is matching connected (which holds since  $\mathbf{cov}(G)$  is 2-extendable and thus also 1-extendable) but not matching 2-connected. Then there is some matching separator  $S = \{x, y\}$  of order 1.

Since  $S$  is a matching separator, there are at least two distinct matching components, say  $C_1$  and  $C_2$ , in  $G - S$  and  $x$  and  $y$  are adjacent to these components. Let  $x_1$  be a neighbour of  $x$  in  $C_1$  and  $y_2$  be a neighbour of  $y$  in  $C_2$ . Since  $\mathbf{cov}(G)$  is 2-extendable, the edges  $xx_1$  and  $yy_2$  are contained in a common perfect matching, say  $M'$ . On the other hand,  $xy$  is contained in a perfect matching of  $G$  as well; let  $M$  be such a matching. Therefore, from Lemma 2.6 there is an  $M$ - $M'$ -alternating cycle  $C$  containing the edge  $xy$  and thus also containing  $xx_1$  and  $yy_2$ . Notice then that every edge of  $C$  is contained in  $\mathbf{cov}(G)$  and thus the edges of  $C - xy$  belong to  $\mathbf{cov}(G) - S$ . Since  $S$  has order 1 and  $\mathbf{cov}(G)$  is 2-extendable, from Lemma 4.3, it follows that  $\mathbf{cov}(G - S) = \mathbf{cov}(G) - S$ . Therefore, all the edges of  $C - xy$  belong to  $\mathbf{cov}(G - S)$ , a contradiction to the assumption that  $C_1$  and  $C_2$  are distinct matching components of  $G - S$ . Thus  $G$  is matching 2-connected.

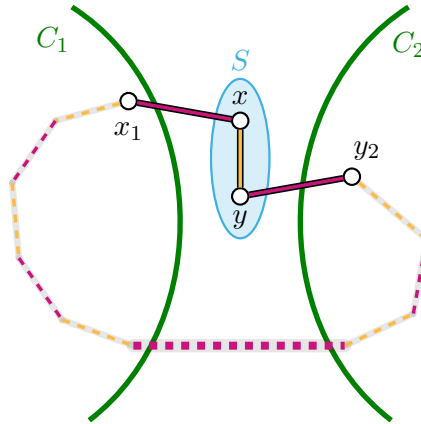


Figure 7: The  $M$ - $M'$ -alternating cycle joining  $C_1$  and  $C_2$ .

Let now  $n > 2$ . Since  $\mathbf{cov}(G)$  is  $n$ -extendable, it is also  $(n - 1)$ -extendable and therefore by induction matching  $(n - 1)$ -connected. Suppose  $G$  is not matching  $n$ -connected. Then  $G$  has a

matching separator  $S$  of order  $n - 1$ . Let  $M$  be a perfect matching of the strongly central set  $S$  and let  $e = xy \in M$  and  $S' = S \setminus \{x, y\}$ . Then  $S'$  is a strongly central set of  $G$  of order  $n - 2$  and hence  $G \setminus S'$  is matching connected. Moreover,  $e$  is a matching separator of  $G - S'$ . However, since  $S'$  is of order  $n - 2$  the graph  $\mathbf{cov}(G) - S'$  is 2-extendable. Furthermore, since the graph  $G$  is  $n$ -extendable it holds that  $\mathbf{cov}(G - S') = \mathbf{cov}(G) - S'$ . Therefore, the graph  $\mathbf{cov}(G - S')$  is also 2-extendable. From the induction hypothesis, it holds that  $G - S'$  is matching 2-connected, a contradiction to  $e$  being a matching separator of  $G - S'$ .  $\square$

Before we start investigating the reverse direction in general, we will show the tightness between matching  $n$ -connectivity and  $n$ -extendability in bipartite graphs. In particular, we obtain the following characterisation of  $n$ -extendable bipartite graphs.

**Lemma 4.5** *A bipartite graph is matching  $n$ -connected for  $n \geq 1$  if and only if  $\mathbf{cov}(G)$  is  $n$ -extendable.*

PROOF. Let  $G$  be a bipartite graph. Note that from Theorem 4.4, if  $\mathbf{cov}(G)$  is  $n$ -extendable then  $G$  is also matching  $n$ -connected. Let us then assume that  $G$  is matching  $n$ -connected. Then, from Lemma 3.3,  $G$  is connected and, from Theorem 3.9,  $\mathcal{D}(G, M)$  is strongly  $n$ -connected for all  $M \in \mathcal{M}(G)$ . Theorem 1.4 asserts that  $\mathbf{cov}(G)$  is  $n$ -extendable and completes the proof of the lemma.  $\square$

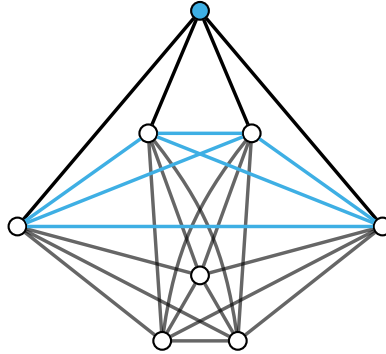


Figure 8: A matching 2-connected graph that is not 2-extendable.

The reverse of Theorem 4.4 does not hold for general graphs. In particular, there are non-bipartite graphs for which the reverse of Theorem 4.4 does not hold. One example for this is the graph in Figure 8. Each of the six blue edges is contained in a perfect matching of the graph, but never two of them at the same time. Otherwise vertex  $v$  on top could not be matched at all.

In the following we will investigate the structure of matching  $n$ -connected graphs which, as the example in Figure 8, are not equally extendable and matching connected. To do this, our main tool will be Theorem 1.2. We aim to find more concrete descriptions of the sets  $S$  and the odd components of  $G - S$  under the given conditions. Recall that  $c_0(G)$  denotes the number of components of a graph  $G$  that have an odd number of vertices.

**Lemma 4.6** *Let  $G$  be a matching connected graph and  $S \subseteq V(G)$  with  $c_0(\mathbf{cov}(G) - S) = |S| \geq 1$ . Then  $\mathbf{cov}(G) - S$  does not contain an even component.*

PROOF. Let  $S \subseteq V(G)$  such that  $c_0(\mathbf{cov}(G) - S) = |S| \geq 1$  and  $C \subseteq \mathbf{cov}(G) - S$  be an even component. Clearly then, since  $c_0(\mathbf{cov}(G) - S) \geq 1$ , there are at least two components of  $G - S$ . Thus, there is at least one edge of  $\mathbf{cov}(G)$  which has no endpoint in  $C$  and is contained in at least one perfect matching of  $\mathbf{cov}(G)$ . Let  $x \in V(C)$  be an arbitrary vertex, and  $e \in E(\mathbf{cov}(G))$  be some edge with no endpoint in  $C$ . Then there is some perfect matching  $M \in \mathcal{M}(G)$  and a strong  $x$ - $e$ - $M$ -extendable path  $P$  by Theorem 2.14. Since  $P$  is a strongly extendable path there exists an edge  $e' = x'y' \in E(P)$  with one endpoint, say  $x'$  in  $C$ , and the other one, say  $y'$ , in  $S$  together with some perfect matching  $M' \in \mathcal{M}(G)$  with  $e' \in M'$ .

Consider the graph  $G' := \mathbf{cov}(G - x' - y')$  with  $S' := S \setminus \{y'\}$ . Clearly  $M' \setminus \{e'\}$  is a perfect matching of  $\mathbf{cov}(G')$ . Note that all odd components of  $\mathbf{cov}(G) - S$  are also odd components of  $\mathbf{cov}(G') - S'$  and there are exactly  $|S|$  of them. Let  $C_1, C_2, \dots, C_l$  be the components of  $\mathbf{cov}(G') - S'$  for which  $C_i \subseteq C - x'$  in  $\mathbf{cov}(G') - S'$ ,  $i \in [l]$ . Since  $C - x'$  has an odd amount of vertices, one of the above components, say  $C_{i_0}$ , is an odd component of  $\mathbf{cov}(G') - S'$ . Thus,  $c_0(\mathbf{cov}(G') - S') \geq |S| + 1$ . However,  $|S'| = |S| - 1$  and thus,  $c_0(\mathbf{cov}(G') - S') \geq |S'| + 2$ , which is a contradiction to Tutte's Theorem (Theorem 1.1).  $\square$

The following two results may be derived from Lemma 3.3. We use our knowledge on ordinary connectivity to narrow down the structure of the odd components we receive from the deletion of some set  $S$  with  $|S| = c_0(\mathbf{cov}(G) - S)$ . The first one follows almost immediately.

**Corollary 4.7** *If  $G$  is a matching  $n$ -connected graph, where  $n \geq 1$ , and  $S \subseteq V(G)$  such that  $|S| = c_0(\mathbf{cov}(G) - S)$ , then  $c_0(\mathbf{cov}(G) - S) = 1$ , or  $c_0(\mathbf{cov}(G) - S) \geq n + 1$ .*

PROOF. Let  $G$  be a matching  $n$ -connected graph and recall that a graph  $G$  is matching  $n$ -connected if and only if  $\mathbf{cov}(G)$  is matching  $n$ -connected. Therefore,  $\mathbf{cov}(G)$  is matching  $n$ -connected and, from Lemma 3.3,  $\mathbf{cov}(G)$  is  $(n + 1)$ -connected. Notice then that if  $|S| < n + 1$  then  $\mathbf{cov}(G) - S$  is connected and thus  $c_0(\mathbf{cov}(G) - S) \leq 1$ . Finally, if  $|S| \geq n + 1$  then  $c_0(\mathbf{cov}(G) - S) \geq n + 1$ .  $\square$

**Corollary 4.8** *If  $G$  is a matching  $n$ -connected graph, where  $n \geq 1$ , and  $S \subseteq V(G)$  such that  $|S| = c_0(\mathbf{cov}(G) - S)$ , then at least one of the following holds:  $S$  is an independent set, every odd component contains at least 3 vertices, or  $c_0(\mathbf{cov}(G) - S) \geq n + 2$ .*

PROOF. Note that, from Corollary 4.7 we obtain that either  $|S| = c_0(\mathbf{cov}(G) - S) \leq 1$  or  $|S| = c_0(\mathbf{cov}(G) - S) \geq n + 1$ . If  $|S| = 1$  then  $S$  is an independent set and the assertion holds. So we may assume that  $|S| = c_0(\mathbf{cov}(G) - S) \geq n + 1$ . If  $c_0(\mathbf{cov}(G) - S) \geq n + 2$  the assertion again holds so let us assume that  $|S| = c_0(\mathbf{cov}(G) - S) = n + 1$ . If each odd component of  $\mathbf{cov}(G) - S$  contains at least 3 vertices then we are done. Suppose that there is at least one trivial odd component, that is, a component containing exactly one vertex  $v$ . Notice that since  $G$  is matching  $n$ -connected then  $\mathbf{cov}(G)$  is matching  $n$ -connected and thus  $\mathbf{cov}(G)$  is matching  $n$ -connected. Hence, from Lemma 3.3,  $\mathbf{cov}(G)$  is  $(n + 1)$ -connected and thus the degree of  $v$  is at least  $n + 1$ . Then  $N(v) = S$  and thus, by Lemma 3.4,  $S$  is an independent set, concluding the proof of the corollary.  $\square$

**Lemma 4.9** *Let  $G$  be a matching connected graph and  $S \subseteq V(G)$  with  $c_0(\mathbf{cov}(G) - S) = |S| \geq 2$ . If there is at least one odd component  $C \subseteq \mathbf{cov}(G) - S$  with  $|V(C)| \geq 3$ ,  $G$  is not matching 2-connected.*



PROOF. Let  $G$  be a counter example. Then there is a component  $C$  in  $\mathbf{cov}(G) - S$  with  $|C| \geq 3$  and  $G$  is matching 2-connected. Let  $y \in V(C)$  such that there is some  $M \in \mathcal{M}(G)$  with  $xy \in M$  and  $x \in S$ . Let  $G' := \mathbf{cov}(G - x - y)$  and  $S' := S \setminus \{x\}$ . Clearly  $G'$  is matching connected and  $M - \{xy\}$  is a perfect matching of  $G'$ . Then by Tutte's Theorem (Theorem 1.1)  $c_0(G' - S') \leq |S'|$ .

Let  $C''$  be any component of  $\mathbf{cov}(G') - S'$ . Then there is some component  $C' \subseteq \mathbf{cov}(G) - S$  with  $C'' \subseteq C'$ . From Lemma 4.6,  $|C'|$  is odd. Thus, for every (odd) component  $C' \neq C$  of  $\mathbf{cov}(G) - S$  there is an odd component in  $\mathbf{cov}(G') - S'$ . Therefore,  $c_0(\mathbf{cov}(G') - S') \geq |S| - 1 = |S'|$  holds and hence  $c_0(\mathbf{cov}(G') - S') = |S'|$ .

Let  $C_1, C_2, \dots, C_l$  be the connected components of  $\mathbf{cov}(G') - S'$  for which  $C_i \subseteq C - x'$  for  $i = 1, \dots, l$ . It follows, from  $c_0(\mathbf{cov}(G') - S') = |S'|$ , that all components  $C_1, C_2, \dots, C_l$  are even, a contradiction to Lemma 4.6.  $\square$

So if  $G$  is a matching 2-connected graph there exists  $S \subseteq V(G)$  with  $c_0(\mathbf{cov}(G) - S) = |S|$ , where  $|S| \geq 2$ , all (odd) components of  $\mathbf{cov}(G) - S$  are isolated vertices. In Figure 8 we have an example of a matching 2-connected graph which is not 2-extendable, but it contains no set  $S$  with the above properties. Since this graph still has to have a witness for it not satisfying the conditions of Theorem 1.2, there must be some set  $S$  whose deletion produces  $|S| - 2$  odd components. An example for such a set is the neighbourhood of the blue vertex.

In this paper, we will investigate the properties of matching  $n + 1$ -connected graphs whose cover graphs are not  $n$ -extendable. We will start with the smallest  $n$  for which this is possible. Hence we consider the case where  $n = 3$  since the cover graph of every matching 2-connected graph is matching covered and connected, hence 1-extendable, by definition.

**Lemma 4.10** *If  $G$  is a matching 3-connected graph such that  $\mathbf{cov}(G)$  is not 2-extendable, then there is some  $S \subseteq V(G)$  with  $c_0(\mathbf{cov}(G) - S) = |S| - 2$  and*

- (i)  $\nu(\mathbf{cov}(G)[S]) \geq 2$ ,
- (ii)  $|E(\mathbf{cov}(G)[S])| \geq 3$ ,
- (iii)  $c_0(\mathbf{cov}(G) - S) \geq 4$ ,
- (iv)  $\mathbf{cov}(G) - S$  has no even components, and
- (v)  $|C| = 1$  for all odd components  $C \subseteq \mathbf{cov}(G) - S$ .

PROOF. Suppose that either there is no set  $S \subseteq V(G)$  with  $c_0(\mathbf{cov}(G) - S) = |S| - 2$  or for every set  $S \subseteq V(G)$  with  $c_0(\mathbf{cov}(G) - S) = |S| - 2$ ,  $\nu(\mathbf{cov}(G)[S']) \leq 1$  holds as well. We are going to invoke Theorem 1.2. From Tutte's Theorem (Theorem 1.1), since  $G$  has a perfect matching, it holds that for every set  $S \subseteq V(G)$ ,  $c_0(\mathbf{cov}(G) - S) \leq |S|$ . Thus, condition 1 of Theorem 1.2 holds. Moreover, condition 1 of Theorem 1.2 holds for  $h = 1$ . Thus, since by the hypothesis  $\mathbf{cov}(G)$  is not 2-extendable, Theorem 1.2 yields the existence of a set  $S' \subseteq V(G)$  with  $c_0(\mathbf{cov}(G) - S') = |S'|$  and  $\nu(\mathbf{cov}(G)[S']) \geq 1$ .

Since  $\nu(\mathbf{cov}(G)[S']) \geq 1$ , it follows that  $|S'| \geq 2$ . Thus, since  $\mathbf{cov}(G)$  is matching 3-connected from Corollary 4.7, we obtain that  $|S'| = c_0(\mathbf{cov}(G) - S') \geq 4$ . Then, Lemma 4.9 asserts that there is no odd component with at least 3 vertices in  $\mathbf{cov}(G) - S'$ . Therefore, by Corollary 4.8, either  $S'$  is an independent set or  $c_0(\mathbf{cov}(G) - S') \geq 5$ . Since  $\nu(\mathbf{cov}(G)[S']) \geq 1$ ,  $S'$  is not an independent set and thus,  $|S'| \geq 5$ .



Let now  $C \subseteq \mathbf{cov}(G) - S'$  be an odd component and  $\{x\} = V(C)$  (since all odd components have exactly one vertex). Set  $S'' := S' \cup \{x\}$ . We obtain that  $|S''| = |S'| + 1$  and  $c_0(\mathbf{cov}(G) - S'') = |S'| - 1$ . Hence  $c_0(\mathbf{cov}(G) - S'') = |S''| - 2$  contradicting the assumption, that there is no set  $S \subseteq V(G)$  with  $c_0(\mathbf{cov}(G) - S) = |S| - 2$ . Moreover, we claim that  $\nu(\mathbf{cov}(G)[S'']) \geq 2$  holds as well. This will provide a contradiction to the assumption that for every set  $S \subseteq V(G)$  with  $c_0(\mathbf{cov}(G) - S) = |S| - 2$ ,  $\nu(\mathbf{cov}(G)[S'']) \leq 1$ .

Notice that if  $\nu(\mathbf{cov}(G)[S']) \geq 2$ , then  $\nu(\mathbf{cov}(G)[S'']) \geq 2$  and the claim holds. Thus, let us assume that  $\nu(\mathbf{cov}(G)[S']) = 1$  and let  $e = pq \in S'$  be the edge in  $S'$  inducing a matching of size 1. Since  $\mathbf{cov}(G)$  is matching 3-connected, from Theorem 3.3,  $\mathbf{cov}(G)$  is also 4-connected. Therefore, the vertex  $x$  has at least 4 neighbours in  $S'$  in  $\mathbf{cov}(G)$ . Let  $y$  be a neighbour of  $x$  in  $S' \setminus \{p, q\}$ . Then notice that the edges  $pq$  and  $xy$  induce a matching of size 2 in  $\mathbf{cov}(G)[S']$  and thus,  $\nu(\mathbf{cov}(G)[S'']) \geq 2$ . Therefore, there is some set  $S \subseteq V(G)$  with  $c_0(\mathbf{cov}(G) - S) = |S| - 2$  and  $\nu(\mathbf{cov}(G)[S]) \geq 2$  and fix this set from now on.

We now prove that all odd components of  $\mathbf{cov}(G) - S$  are trivial. By definition, every edge in  $\mathbf{cov}(G)[S]$  is contained in a perfect matching of  $G$ . Moreover, since  $\nu(\mathbf{cov}(G)[S]) \geq 2$ , there exist at least two edges in  $\mathbf{cov}(G)[S]$ . Let  $pq \in S$  and let  $M \in \mathcal{M}(G)$  with  $pq \in M$ . From the fact that  $G$  is matching 3-connected, the graph  $G' := \mathbf{cov}(G - x - y)$  is matching 2-connected. Let  $S' := S \setminus \{x, y\}$ . Observe and note that all odd components of  $\mathbf{cov}(G) - S$  are also odd components of  $\mathbf{cov}(G') - S'$ . Thus,  $c_0(\mathbf{cov}(G') - S') \geq c_0(\mathbf{cov}(G) - S) = |S| - 2 = |S'|$ . Observe also that  $M \setminus pq$  is a perfect matching of  $G'$  and thus of  $\mathbf{cov}(G')$ . Therefore, by Tutte's Theorem (Theorem 1.1),  $c_0(\mathbf{cov}(G') - S') = |S'|$ . Finally, notice that  $|S| \geq 4$  (since  $\nu(\mathbf{cov}(G)[S]) \geq 2$ ) and thus,  $|S'| \geq 2$ . Since  $G'$  is matching 2-connected,  $c_0(\mathbf{cov}(G') - S') = |S'|$ , and  $|S'| \geq 2$ , from Lemma 4.9 we obtain that all of the odd components of  $\mathbf{cov}(G') \setminus S'$ , and thus of  $\mathbf{cov}(G) - S$ , are trivial.

Next, we prove that there are at least 4 odd components in  $\mathbf{cov}(G) - S$ . For this observe first that, from Corollary 4.7, since  $G'$  is matching 2-connected,  $c_0(\mathbf{cov}(G') - S') = |S'|$ , and  $|S'| \geq 2$ , we derive that  $|S'| \geq 3$  and therefore,  $|S| \geq 5$ . Notice that if  $|S| \geq 6$  then  $c_0(\mathbf{cov}(G) - S) = |S| - 2 \geq 4$  and the assertion holds, so suppose  $|S| = 5$ . Notice that  $|V(G)| = |V(G')| + 2$ . Again,  $G'$  is matching 2-connected and  $c_0(\mathbf{cov}(G') - S') = |S'|$ ,  $\mathbf{cov}(G') - S'$  does not have any even components. Thus,  $|V(G')| = |S'| + c_0(\mathbf{cov}(G') - S') = 2|S'|$ . Therefore,  $|V(G)| = 2|S'| + 2$ . Since  $|S| = 5$  we have that  $|S'| = 3$  and  $|V(G)| = 8$ . By definition of matching 3-connectivity,  $G$  has at least 10 vertices, a contradiction and thus  $|S| \geq 6$  and  $\mathbf{cov}(G) - S$  has at least 4 odd components.

We now prove that  $\mathbf{cov}(G) - S$  has no even components. Recall first that the graph  $G'$  has no even component and all its odd components are trivial, and thus,  $|V(G')| = |S'| + c_0(\mathbf{cov}(G') - S')$ . Moreover, recall that  $|V(G)| = |V(G')| + 2$ . Finally, recall that,  $c_0(\mathbf{cov}(G) - S) = |S| - 2 = |S'| = c_0(\mathbf{cov}(G') - S')$ . Therefore,  $|V(G)| = |S'| + c_0(\mathbf{cov}(G') - S') + 2 = c_0(\mathbf{cov}(G) - S) + |S|$ . This implies that  $\mathbf{cov}(G) - S$  has no even components.

To finish the proof, it remains to show that  $|E(\mathbf{cov}(G)[S])| \geq 3$ . Again aiming for a contradiction we assume  $|E(\mathbf{cov}(G)[S])| = 2$ . Then let  $uv$  and  $xy$  be the two edges of  $\mathbf{cov}(G)[S]$  and recall that  $\mathbf{cov}(G)[S]$  contains a matching of size 2. Therefore the edges  $uv$  and  $xy$  are disjoint. Consider again  $G'$  and  $S'$ . Let  $M \in \mathcal{M}(G)$  with  $xy \in M$ , Then  $M' \setminus \{xy\}$  is a perfect matching of  $\mathbf{cov}(G')$ . Moreover,  $c_0(\mathbf{cov}(G') - S') = |S'|$ . Therefore, since  $\mathbf{cov}(G')$  has a perfect matching, each vertex of  $S'$  has to be matched to a vertex of a distinct (odd) component  $C$ . Notice then that  $uv \notin E(\mathbf{cov}(G'))$  and  $u$  and  $v$  must be matched to two distinct components of  $\mathbf{cov}(G') - S'$  for all  $M \in \mathcal{M}(G')$ . Fix some  $M \in \mathcal{M}(G')$  and let  $uu', vv' \in M$  be the two matching edges containing  $u$  and  $v$ . Consider  $G'' := \mathbf{cov}(G - \{u, u', v, v'\})$  and  $S'' := S' \setminus \{u, v\}$ . Notice then that all odd components of  $\mathbf{cov}(G) - S$

(but the ones consisting of  $u'$  and  $v'$ ) are also odd components of  $\mathbf{cov}(G'') - S''$ . Therefore, since  $G$  does not have any even components,  $c_0(G'' - S'') = (|S| - 2) - 2 = |S''| - 2$ . Notice also that since  $S'' = S \setminus \{u, v\}$  then  $\nu(G''[S'']) = 1$ . Since every component of  $G'' - S''$  is an isolated vertex and there are  $|S''| - 2$  of them, every perfect matching of  $G''$  has to contain at least one edge of  $G''[S'']$ . But since  $\{xy\} = E(G''[S''])$ ,  $xy$  has to be contained in every perfect matching of  $G''$ . Hence  $\{u, u', v, v'\}$  is a matching separator of  $G$  contradicting its matching 3-connectivity. Hence  $|E(\mathbf{cov}(G)[S])| \geq 3$ .  $\square$

Using Lemma 4.10 as a base, we are now able to define a more general class of graphs that has high matching connectivity, but is not equally extendable. These graphs have a very interesting property, which links non-bipartite graphs to bipartite ones and emphasises the importance of bipartite structures in matching theory.

**Definition 4.11** Let  $h \geq 1$  be an integer. A graph  $G$  is called *brace  $h$ -critical*, if  $G$  is matching  $(h + 2)$ -connected,  $\mathbf{cov}(G)$  is  $h$ -extendable and  $V(G) = S \cup I$  with

- (i)  $S \cap I = \emptyset$ ,
- (ii)  $|S| = |I| + 2h$ ,
- (iii)  $I$  is an independent set in  $\mathbf{cov}(G)$ ,
- (iv)  $\nu(\mathbf{cov}(G)[S]) \geq h + 1$ , and
- (v) if  $M \subseteq E(\mathbf{cov}(G)[S])$  is a matching with  $|M| = h$  and  $S_M := \bigcup_{e \in M} e$ , then  $\mathbf{cov}(G - S_M)$  is a matching 2-connected bipartite graph with colour classes  $I$  and  $S \setminus S_M$ .

If  $h = 1$ , we call  $G$  *brace critical*.

**Lemma 4.12** Let  $G$  be a matching 3-connected graph. Then  $\mathbf{cov}(G)$  is not 2-extendable if and only if  $G$  is brace critical.

PROOF. If  $G$  is brace critical, we have  $V(G) = S \cup I$  with  $|S| = |I| + 2$ . Since  $I$  is an independent set in the cover graph and  $\mathbf{cov}(G) - S = G[I]$ , we have  $c_0(\mathbf{cov}(G) - S) = |I| = |S| - 2$ , but  $\nu(\mathbf{cov}(G)[S]) \geq 2$  (from the definition of brace critical graphs). Hence by Theorem 1.2,  $\mathbf{cov}(G)$  is not 2-extendable.

Now suppose  $G$  is matching 3-connected, but  $\mathbf{cov}(G)$  is not 2-extendable. Notice that since  $G$  is matching 3-connected then it is also matching connected. Hence  $\mathbf{cov}(G)$  is matching connected and thus, by Theorem 4.2,  $\mathbf{cov}(G)$  is also 1-extendable. Moreover, by Lemma 4.10, there is some  $S \subseteq V(G)$  with

- $c_0(\mathbf{cov}(G) - S) = |S| - 2$ ,
- $\nu(\mathbf{cov}(G)[S]) \geq 2$ ,
- $\mathbf{cov}(G) - S$  has no even components, and
- $|C| = 1$  for all odd components  $C \subseteq \mathbf{cov}(G) - S$ .

Notice then that the set  $I := V(\mathbf{cov}(G) - S)$  is an independent set in  $\mathbf{cov}(G)$ . Furthermore, observe that since  $G$  has a perfect matching and  $|I| = |S| - 2$  no perfect matching of  $G$  can contain more than one edge of  $\mathbf{cov}(G)[S]$  (as otherwise there would be vertices of  $I$  not covered by such a matching). Notice then that for  $\mathbf{cov}(G - x - y)$  and  $S' = S \setminus \{x, y\}$  it holds that  $c_0(\mathbf{cov}(G') - S') = |S'|$ . This implies that  $\mathbf{cov}(G - x - y)$  is bipartite for all  $xy \in E(\mathbf{cov}(G)[S])$ . In particular, the colour classes are  $I$  and  $S \setminus \{x, y\}$ . In addition, since  $G$  is matching 3-connected,  $\mathbf{cov}(G - x - y)$  is matching 2-connected. Thus all conditions necessary for  $G$  to be brace critical are satisfied.  $\square$

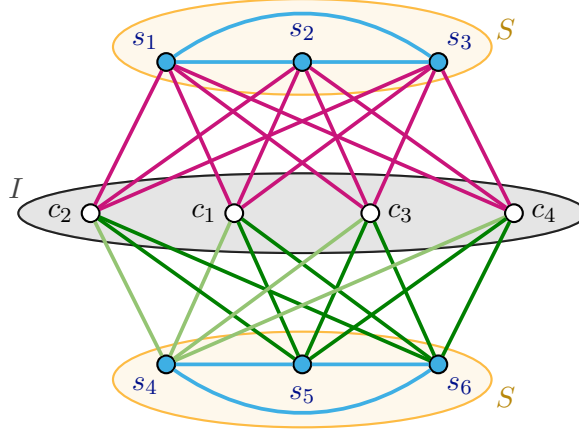


Figure 9: A brace critical graph.

The above lemma asserts, that the brace critical graphs are exactly the ones described in Lemma 4.10. Figure 9 shows an example of such a graph. Note also that, since  $\mathbf{cov}(G - S_M)$  is a matching 2-connected bipartite graph for every matching  $M \subseteq E(\mathbf{cov}(G)[S])$  of size  $h$ , then by Lemma 4.5, it is also 2-extendable. Thus, for every  $S_M$  we obtain a brace.

We now have obtained all the pieces we need in order to state and prove the main result of this section.

**Theorem 4.13** *Let  $n \geq 1$  be an integer and  $G$  be a matching  $(n + 1)$ -connected graph. Then  $\mathbf{cov}(G)$  either is  $n$ -extendable, or  $G$  is brace  $h$ -critical for some  $1 \leq h \leq n - 1$ .*

PROOF. We will prove the assertion by induction over  $n$ . For  $n = 1$ ,  $G$  is matching 2-connected and thus from Theorem 4.2  $\mathbf{cov}(G)$  is 1-extendable.

So let  $n \geq 2$  and  $G$  be matching  $(n + 1)$ -connected. In particular,  $G$  is matching  $n$ -connected. Therefore, by the induction hypothesis,  $G$  is either brace  $h$ -critical for some  $1 \leq h \leq n - 2$ , or  $\mathbf{cov}(G)$  is  $(n - 1)$ -extendable. If  $G$  is brace  $h$ -critical for some  $1 \leq h \leq n - 2$  then we are done.

Suppose then that  $\mathbf{cov}(G)$  is  $(n - 1)$ -extendable, but not  $n$ -extendable. Our aim is to show that  $\mathbf{cov}(G)$  is brace  $(n - 1)$ -critical. Since  $G$  has a perfect matching, from Tutte's Theorem (Theorem 1.1), for all  $S \subseteq V(G)$ ,  $c_0(\mathbf{cov}(G) - S) \leq |S|$ . Then, since  $\mathbf{cov}(G)$  is not  $n$ -extendable, Theorem 1.2 provides us with the existence of some set  $S \subseteq V(G)$  with  $c_0(\mathbf{cov}(G) - S) = |S| - 2j$  with  $0 \leq j \leq n - 1$  and  $\nu(\mathbf{cov}(G)[S]) \geq j + 1$ . However, since  $\mathbf{cov}(G)$  is  $(n - 1)$ -extendable, the same theorem asserts  $\nu(\mathbf{cov}(G)[S]) \leq j$  for all sets  $S \subseteq V(G)$  with  $c_0(\mathbf{cov}(G) - S) = |S| - 2j$ ,

$1 \leq j \leq n - 2$ . Therefore, there exists some  $S \subseteq V(G)$  with  $c_0(\mathbf{cov}(G) - S) = |S| - 2(n - 1)$  and  $\nu(\mathbf{cov}(G)[S]) \geq n$ .

Let then  $M_S$  be a matching of  $\mathbf{cov}(G)[S]$  of size  $n$  and  $M_{n-2} \subseteq M_S$  be any matching of size  $n - 2$  in  $\mathbf{cov}(G)[S]$  that is contained in  $M_S$ . Moreover, let  $S_{n-2} := \bigcup_{e \in M_{n-2}} e$  be the set of the endpoints of the edges in  $M_{n-2}$  and  $G' := G - S_{n-2}$ . By Lemma 4.3, we obtain that  $\mathbf{cov}(G - S_{n-2}) = \mathbf{cov}(G) - S_{n-2}$ . Thus, it also holds that  $\mathbf{cov}(G') = \mathbf{cov}(G) - S_{n-2}$ . Notice that since  $G$  is matching  $(n + 1)$ -connected then  $G'$  is matching 3-connected.

Since  $M_{n-2}$  is a matching of size  $n - 2$  in  $\mathbf{cov}(G)$  and  $\mathbf{cov}(G)$  is  $(n - 1)$ -extendable, there exists a perfect matching  $M$  of  $\mathbf{cov}(G)$  with  $M_{n-2} \subseteq M$ . Moreover, the matching  $M \setminus M_{n-2}$ , is a perfect matching of  $\mathbf{cov}(G')$ . Thus, for all  $X \subseteq V(G')$ , it holds that  $c_0(\mathbf{cov}(G') - X) \leq |X|$ . Let  $S' = S \setminus S_{n-2}$ . Notice that for the set  $S'$ ,  $\nu(\mathbf{cov}(G')[S']) \geq 2$  and  $|S'| = |S| - 2(n - 2)$ . Moreover,  $\mathbf{cov}(G') - S' = \mathbf{cov}(G) - S_{n-2} - S' = \mathbf{cov}(G) - S$ . Thus,  $c_0(\mathbf{cov}(G') - S') = c_0(\mathbf{cov}(G) - S)$ . Recall that  $c_0(\mathbf{cov}(G) - S) = |S| - 2(n - 1)$ . Therefore,  $c_0(\mathbf{cov}(G') - S') = |S'| + 2(n - 2) - 2(n - 1) = |S'| - 2$ . Since we also have that  $\nu(\mathbf{cov}(G')[S']) \geq 2$ , from Theorem 1.2, we obtain that  $\mathbf{cov}(G')$  is not 2-extendable. Since  $G'$  is also matching 3-connected, by Lemma 4.12,  $G'$  is brace critical with

$V(G') = S' \cup I'$ . Now let  $I := I'$ .

It what remains we show that  $G$  with  $V(G) = S \cup I$  is brace  $(n - 1)$ -critical. We already know that

$$c_0(\mathbf{cov}(G) - S) = |S| - 2(n - 1)$$

and

$$|I| = |S'| - 2 = |S'| - 2 + |S_{n-2}| - |S_{n-2}| = |S| - 2(n - 1).$$

Hence  $V(G) \setminus S = I$  is an independent set in  $\mathbf{cov}(G)$  as well. Let  $M_{n-1} \subseteq E(\mathbf{cov}(G)[S])$  be a matching of size  $n - 1$  and  $S_{n-1} := \bigcup_{e \in M_{n-1}} e$ . Then it is easy to see that  $\mathbf{cov}(G - S_{n-1})$  is bipartite and matching 2-connected with colour classes  $I$  and  $S \setminus S_{n-1}$ .  $\square$

## References

- [1] Jack Edmonds, WR Pulleyblank, and L Lovász. Brick decompositions and the matching rank of graphs. *Combinatorica*, 2(3):247–274, 1982.
- [2] László Lovász. Matching structure and the matching lattice. *Journal of Combinatorial Theory, Series B*, 43(2):187–222, 1987.
- [3] László Lovász and Michael D Plummer. Matching theory. number 29 in annals of discrete mathematics, 1986.
- [4] Denis Naddef and William R Pulleyblank. Matchings in regular graphs. *Discrete Mathematics*, 34(3):283–291, 1981.
- [5] MD Plummer. Matching extension in regular graphs. Technical report, DTIC Document, 1989.

- [6] Michael D Plummer. On  $n$ -extendable graphs. *Discrete Mathematics*, 31(2):201–210, 1980.
- [7] Michael D Plummer. Extending matchings in graphs: a survey. *Discrete Mathematics*, 127(1):277–292, 1994.
- [8] Michael D Plummer. Recent progress in matching extension. In *Building Bridges*, pages 427–454. Springer, 2008.
- [9] Yu Qinglin. Characterizations of various matching extensions in graphs. *Australasian Journal of Combinatorics*, 2:55–64, 1993.
- [10] Neil Robertson, Paul D Seymour, and Robin Thomas. Permanents, pfaffian orientations, and even directed circuits. *Annals of Mathematics*, 150(3):929–975, 1999.
- [11] William T Tutte. The factorization of linear graphs. *Journal of the London Mathematical Society*, 1(2):107–111, 1947.
- [12] Fuji Zhang and Heping Zhang. Construction for bicritical graphs and  $k$ -extendable bipartite graphs. *Discrete mathematics*, 306(13):1415–1423, 2006.